MIDTERM: 120 Minutes

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
100	20	20	20	20	20	200

1 **Circle one answer per question.** 10 **points for each correct answer.**

(a) Compute the sum $\sum_{n=1}^{4} 3^n$.

$$\sum_{n=1}^{4} 3^n = \underbrace{\sum_{n=0}^{4} 3^n}_{\text{common sum}} - 1 = \frac{1-3^5}{1-3} - 1 = 120$$

- A 120.
- B 121.
- C 242.
- D 243.
- E None of the above.

(b) What is the last digit of 3^{11}?

$$3^{11} \bmod 10 = 3^{4\cdot 2+3} \bmod 10 = 81^2 \cdot 27 \bmod 10$$
$$= \left[(81 \bmod 10)^2 \cdot 27 \bmod 10\right] \bmod 10$$
$$= 27 \bmod 10 = 7$$

- A 1.
- B 3.
- C 7.
- D 9.
- E None of the above.

(c) A graph has degree sequence $[6, 6, 3, 3, 3, 2, 2]$. How many edges does this graph have?

$\sum \delta_i = 25 \neq 2|E|$ for any E, so Handshaking Thm implies this is not a degree sequence

- A 12.
- B 25.
- C 30.
- D Not enough information to say.
- E Such a graph does not exist.

(d) Suppose a connected planar graph has 18 vertices, each of degree 3. Into how many regions does any planar representation of this graph split the plane?

$$F + V - E = 2 \Rightarrow F + 18 - E = 2 \Rightarrow F = E - 16$$
$$\sum \delta_i = 3 \cdot 18 = 2 \cdot E \Rightarrow E = 27$$
$$\Rightarrow F = 11$$

- A 6.
- B 11.
- C 27.
- D 40.
- E None of the above.

(e) Compute $102^{1211} \bmod 5$.

$$102^{1211} \bmod 5 = (102 \bmod 5)^{1211} \bmod 5$$
$$= 2^{1211} \bmod 5$$
$$= 2^{4\cdot 302+3} \bmod 5 = (16^{302} \bmod 5)\cdot(8 \bmod 5) \bmod 5$$
$$= (16 \bmod 5)^{302} \cdot 3 \bmod 5 = 3$$

$$\begin{array}{r} 302 \\ 4\,\overline{)1211} \\ -12 \\ \hline 00 \\ 11 \\ -8 \\ \hline 3 \end{array}$$

- A 0
- B 1
- C 2
- D 3
- E 4

(f) Which of the following numbers evenly divides $102^{1211} - 3^{1211}$?

[A] 5

[B] 17

[C] 2

[D] 99

[E] None of the above

$102 \equiv 3 \bmod 99 \Rightarrow 102^{1211} \equiv 3^{1211} \bmod 99$

$\Rightarrow 99 \mid 102^{1211} - 3^{1211}$

$\ell \rightarrow g \vee s$

(g) The negation of "If Lassie vomits then she ate grass or she is sick" is:

[A] If Lassie didn't eat grass and is healthy, she will not vomit.

[B] Lassie vomited and did not eat grass and is not sick.

[C] When Lassie eats grass or is sick, she does not vomit.

[D] Lassie did not vomit and she ate grass and is sick.

[E] None of the above.

$\ell \to g \vee s$ equiv to $\neg \ell \vee (g \vee s)$

so $\neg(\ell \to g \vee s)$ equiv to

$\ell \wedge \neg(g \vee s)$

$\equiv \ell \wedge (\neg g \wedge \neg s)$

(h) Which claim below is true?

[A] If $x, y \in \mathbb{Q}$ then $y^x \in \mathbb{Q}$.

[B] x is odd if and only if $x^2 - 1$ is divisible by 8.

[C] If p is prime, then $k^p - k$ is not divisible by p, for any integer k.

[D] None of these claims are true.

[E] All of these claims are true.

x odd $\Rightarrow x = 2n+1 \Rightarrow x^2 = 4n^2 + 4n + 1$

$\Rightarrow x^2 - 1 = 4(n^2 + n)$

and we know $n \in \mathbb{Z} \Rightarrow n^2 + n$ is even, so $8 \mid x^2 - 1$

and $8 \mid x^2 - 1 \Rightarrow x^2 = 8k + 1$ for some k

$\Rightarrow x^2$ odd $\Rightarrow x$ odd, so

x odd $\Leftrightarrow 8 \mid x^2 - 1$

(i) Which of the following asymptotic relationships is correct?

[A] $(n+1)! \in O(n!)$.

[B] $(n+1)! \in \omega(n!)$.

[C] $(n+1)! \in o(n!)$.

[D] $(n+1)! \in \Theta(n!)$.

[E] None of the above.

$(n+1)! = (n+1)\,n!$ so for any $c > 0$,

if $n > c-1$, then $(n+1)! > c\,n!$

so by defn, $(n+1)! \in \omega(n!)$

(j) Which of the following recursions defines a sequence T_n satisfying $T_n \in \Theta(2^n)$?

[A] $T_1 = 2; T_n = T_{n-1}^2$ for $n > 1$.

[B] $T_1 = 2; T_n = 2 + 2T_{n-1}$ for $n > 1$.

[C] $T_1 = 2; T_n = 2nT_{n-1}$ for $n > 1$.

[D] All of the above.

[E] None of the above.

solve by induction: $T_n = 2(2^n - 1)$

Then

$1 \cdot 2^n < T_n < 2 \cdot 2^n$ when $n > 2$

so $T_n = \Theta(2^n)$

2 **Let p be prime. Consider an integer $b \in [1, p-1]$. Use Bezout's Theorem to show that there exists an integer $x \in [1, p-1]$ that satisfies $bx \equiv 1 \bmod p$.**

Prf

Since p is prime and $p \nmid b$, we have $\gcd(p, b) = 1$.
By Bezout's thm, there exist integers $\tilde{x}$ and y satisfying

$$1 = b\tilde{x} + py.$$

Thus $b\tilde{x} \equiv 1 \bmod p$.

By the rules of modular arithmetic,

$$\begin{aligned} b\tilde{x} &\equiv (b \bmod p)(\tilde{x} \bmod p) \bmod p \\ &= b \cdot (\tilde{x} \bmod p) \bmod p, \end{aligned}$$

so letting $x = \tilde{x} \bmod p$, we see that

$$bx \equiv 1 \bmod p.$$

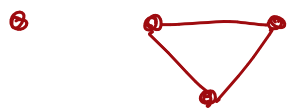
By definition, x is an integer in $[0, p-1]$; the fact that $bx \not\equiv 0 \bmod p$ tells us that $x \neq 0$.
Therefore $x \in [1, p-1]$ is an integer that satisfies

$$bx \equiv 1 \bmod p$$

$\blacksquare$

3 **Prove or disprove: every graph with n vertices and $n-1$ edges is a tree.**

This is false. Here is a counterexample:



This graph has 4 vertices and 3 edges, but contains a cycle and is not connected, and is therefore not a tree

∎

4 **For any positive integer k, prove that $1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$.**

Prf

Direct, using the Integration Method.

From the lower bound integration method,

$$\sum_{i=1}^{n} i^k \geqslant \int_0^n x^k\,dx = \frac{n^{k+1}}{k+1} = c \cdot n^{k+1} \text{ where } c = \frac{1}{k+1}.$$

From the upper bound integration method,

$$\sum_{i=1}^{n} i^k \leq \int_1^{n+1} x^k\,dx = \frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1}$$

$$< \frac{(n+1)^{k+1}}{k+1}$$

$$< \frac{(2n)^{k+1}}{k+1} \quad \text{when } n > 1$$

$$= C \cdot n^{k+1} \text{ where } C = \frac{2^{k+1}}{k+1}.$$

This shows that when $n > 1$,

$$c\,n^{k+1} \leqslant \sum_{i=1}^{n} i^k \leq C\,n^{k+1} \text{ for some } C, c > 0,$$

so by definition

$$\sum_{i=1}^{n} i^k = \Theta\left(n^{k+1}\right)$$

5 **Let** $A_n = \underbrace{1\cdots1}_{n \text{ ones}}$ **for** $n \geq 1$**. Notice that** $A_n = 10A_{n-1} + 1$ **for** $n \geq 2$**. Use induction to show that** $A_n \equiv 3 \bmod 4$ **when** $n \geq 2$**.**

Prf

By induction.

Base case $A_2 = 11 = 8+3$, so $A_2 \equiv 3 \bmod 4$

Inductive Step Assume $A_{n-1} \equiv 3 \bmod 4$, then

$$A_n \bmod 4 = (10 A_{n-1} + 1) \bmod 4$$

$$= \{(10 \bmod 4) \cdot (A_{n-1} \bmod 4) \bmod 4 + 1\} \bmod 4$$

$$= \{2 \cdot 3 \bmod 4 + 1\} \bmod 4$$

$$= 3.$$

By the principle of induction, $\forall n \geq 2$: $A_n \bmod 4 = 3$

$\blacksquare$

6 **Determine the type of proof, and prove: every odd natural number is the difference of two perfect squares.**

Proof

Direct. Let $x = 2n+1$ be an odd natural number, then

$$x = (n+1)^2 - n^2$$

so x is the difference of two perfect squares

$\blacksquare$

SCRATCH

SCRATCH