FINAL: 180 Minutes

Sample Solution

Answer **ALL** questions.
OPEN BOOK (notes, assignments, and textbook) and electronic devices allowed.
NO COLLABORATION or Internet use. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
200	**30**	**30**	**30**	**30**	**30**	**350**

1 Circle one answer per question. 10 points for each correct answer.

(1) In how many ways can six indistinguishable balls be distributed into nine distinguishable bins if each bin can hold up to six balls?

A 54

B 2000

C 3003

D 84

E None of the above

$$\binom{9+6-1}{6} = \binom{10}{6} = \boxed{3003}$$

(2) If n is a positive integer, then how many nonempty bit strings with length not exceeding n consist entirely of ones?

$= \{1, 1\cdot 2, \dots, 1\cdot n\}$

A n+1

B n-1

C n

D 1

E None of the above

(3) A box contains a dozen brown socks and a dozen black socks, all unmatched. If a person takes out socks randomly without replacement, how many socks must she take out to be sure that she has at least two socks of the same color?

pigeonhole principle: there are only two types of socks so after seeing three, at least two of them are the same type

A 2

B 1

C 4

D 3

E None of the above

(4) You are given a deck of 52 cards. You draw from the deck uniformly at random and put the card back. You continue this procedure until you have seen all four aces. What is the expected value of the number of draws you take before seeing all the aces?

$X_1 \sim \text{Geom}\left(\frac{4}{52}\right) \Rightarrow EX_1 = \frac{52}{4}$

$X_2 \sim \text{Geom}\left(\frac{3}{52}\right)$

$X_3 \sim \text{Geom}\left(\frac{2}{52}\right)$

$X_4 \sim \text{Geom}\left(\frac{1}{52}\right) \Rightarrow EX_4 = 52$

$\Rightarrow E\,\text{draws} = 52\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \boxed{108.33}$

A 100.06

B 52

C 61.5

D 108.33

E None of the above

(5) At a birthday party, three types of fruit juices are served to the guests. The supply of juice bottles consists of eight orange juices, 10 apple juices, and 12 fruit punches. If six bottles are randomly selected, what is the probability that all of them are the same variety?

$$\mathbb{P}(\text{all same}) = \mathbb{P}(\text{all OJ}) + \mathbb{P}(\text{all apple}) + \mathbb{P}(\text{all punch})$$
$$= \frac{\binom{8}{6} + \binom{10}{6} + \binom{12}{6}}{\binom{30}{6}} = \boxed{0.002}$$

A 0.14

B 0.01

C 0.0056

D 0.002

E None of the above

(6) What is the probability that, in a group of n people, at least two were born in the same month of the year? You may assume $0 < n \leq 12$.

A $\frac{11}{12} \cdot \frac{10}{12} \cdots \frac{13-n}{12}$

B $\frac{13-n}{12}$

C $1 - \frac{n}{12}$

D $1 - \frac{11}{12} \cdot \frac{10}{12} \cdots \frac{13-n}{12}$

E None of the above

$$\mathbb{P}(\text{at least 2 born in same month}) = 1 - \mathbb{P}(\text{none share month})$$
$$= 1 - \frac{\binom{12}{n} n!}{12^n} = 1 - \frac{\frac{12!}{(12-n)!}}{12^n} = \boxed{1 - \frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \ldots \cdot \frac{12-(n-1)}{12}}$$

(7) Which of the following pairs satisfy $f \in o(g)$?

(I) $f(n) = n^6$, $g(n) = 10n^4$

(II) $f(n) = 5n^2 + 1$, $g(n) = n^2 \log n$

(III) $f(n) = n^{\log n}$, $g(n) = (\log n)^n$

(IV) $f(n) = \cos^2(n\pi)$, $g(n) = \sin^2(n\pi)$

$\frac{f(n)}{g(n)} \to \infty$ as $n \to \infty$ ✗

$\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$ ✓

$\frac{f(n)}{g(n)}$ undefined b/c $f = 1$, $g = 0$

A I

B II, IV

C II, III

D I, II, III

E II, III, IV

(8) In a nearby gas station, 40% of the customers use regular gas, 35% use plus gas and 25% use premium gas. Of those customers using regular gas, only 30% fill their tanks; 60% of plus customers fill their tanks, and 50% of premium customers fill their tanks. If the next customer fills their tank, what is the probability that they used premium gas?

A 0.275

B 0.462

C 0.125

D 0.210

E None of the above

$$\mathbb{P}(\text{premium} \mid \text{fill}) = \frac{\mathbb{P}(\text{fill} \mid \text{premium})\mathbb{P}(\text{prem})}{\mathbb{P}(\text{fill})} = \frac{.25 \times .5}{.455} = \boxed{0.275}$$

$$\mathbb{P}(\text{fill}) = \mathbb{P}(\text{fill} \mid \text{prem})\mathbb{P}(\text{prem}) + \mathbb{P}(\text{fill} \mid \text{plus})\mathbb{P}(\text{plus}) + \mathbb{P}(\text{fill} \mid \text{reg})\mathbb{P}(\text{reg}) = .25 \times .5 + .35 \times .6 + .4 \times .3$$
$$= 0.455$$

(9) What is $7^{21} \bmod 43$?

$7^{21} \bmod 43 = 7^{2\cdot10+1} \bmod 43 = 49^{10}\cdot 7 \bmod 43 = 6^{10}\cdot 7 \bmod 43$
$= 6^{3\cdot3+1}\cdot 7 \bmod 43 = 216^3 \cdot 6 \cdot 7 \bmod 43$
$= 6\cdot 7 \bmod 43$
$= \boxed{42}$

A 1

B 6

C 11

D 36

E 42

(10) If a graph on n vertices is disconnected, what is the largest number of edges it can have?

maximize number of edges by having one isolated vertex and use the remaining $n-1$ to form K_{n-1} > has $\boxed{\binom{n-1}{2} \text{ edges}}$

A $n-1$

B n

C $\binom{n-1}{2}$

D $\binom{n-1}{2} - 1$

E $\binom{n}{2} - 1$

(11) In a clinic, 8% of the patients are infected with a virus. When a blood test is given for this virus, 98% of the infected patients test positive and 3% of the uninfected test positive. What is the probability that a patient who tests negative is infected?

$\mathbb{P}(\text{inf}\mid\text{neg}) = \dfrac{\mathbb{P}(\text{neg}\mid\text{inf})\,\mathbb{P}(\text{inf})}{\mathbb{P}(\text{neg})} = \dfrac{.02\times.08}{0.894} = \boxed{0.002}$

$\mathbb{P}(\text{neg}) = \mathbb{P}(\text{neg}\mid\text{inf})\mathbb{P}(\text{inf}) + \mathbb{P}(\text{neg}\mid\text{not inf})\mathbb{P}(\text{not inf})$
$= .02 \times .08 + 0.97 \times 0.92 = 0.894$

A 0.740

B 0.260

C 0.002

D 0.998

E None of the above

(12) What is the expected value of the sum of numbers appearing on two fair dice, given that the sum of these numbers is at least 9?

$d_1+d_2=9$	$d_1+d_2=10$	$d_1+d_2=11$	$d_1+d_2=12$
3 6	4 6	5 6	6 6
4 5	5 5	6 5	
5 4	6 4		
6 3			

$\Rightarrow \mathbb{E}[d_1+d_2 \mid d_1+d_2 \geq 9] = \frac{4}{10}\cdot 9 + \frac{3}{10}\cdot 10 + \frac{2}{10}\cdot 11 + \frac{1}{10}\cdot 12$
$= \boxed{10}$

A 9

B 10

C 12

D 6

E 8

(13) Which of the following implies $\mathcal{L} \in \mathcal{P}$?

(I) $\mathcal{L}$ is regular — can decide in linear time w/ DFA

(II) $\mathcal{L}$ is Turing decidable

(III) $\overline{\mathcal{L}} \in \mathcal{P}$ — can turn poly-time decider for $\overline{\mathcal{L}}$ into one for $\mathcal{L}$ by inverting output

A I

B II

C III

D I, II

E I, III

(14) $\mathcal{L} = \{x \mid x \text{ is a binary expansion of } \sqrt{2}\}$ can be solved by:

(I) DFA
(II) CFG
(III) Turing Machine

this is an infinite binary string (b/c $\sqrt{2}$ is irrational), so $\mathcal{L}$ is not a language

A all

B I,III

C II,III

D III

E none

(15) Evaluate the sum $T(n) = \sum_{i=1}^{n} \left[\left(\frac{1}{2}\right)^i + \sum_{j=1}^{n} j \right]$.

A $\frac{1}{2}\left(n^3 + n^2 + 4 - \frac{1}{2^{n-1}}\right)$

B $\frac{1}{2}\left(n^3 + n^2 + 4 - \frac{1}{2^{n}}\right)$

C $\frac{1}{2}\left(n^3 + n^2 + 2 - \frac{1}{2^{n}}\right)$

D $\frac{1}{2}\left(n^3 + n^2 + 2 - \frac{1}{2^{n-1}}\right)$

E $\frac{1}{2}\left(n^3 + 2n^2 + n + 4 - \frac{1}{2^{n-1}}\right)$

$T(n) = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i + n \sum_{j=1}^{n} j$

$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-1/2} - 1 + n\left(\frac{n \cdot (n+1)}{2}\right)$

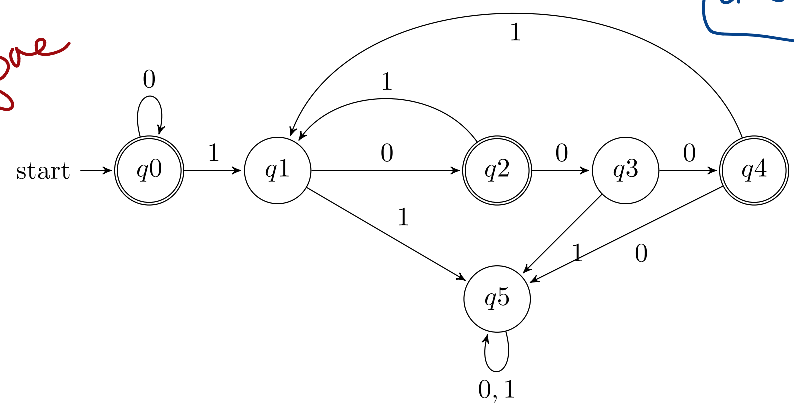
$= 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) - 1 + \frac{n^3}{2} + \frac{n^2}{2}$

$= 1 - \frac{1}{2^n} + \frac{n^3}{2} + \frac{n^2}{2}$

$= \boxed{\frac{1}{2}\left(n^3 + n^2 - \frac{1}{2^{n-1}} + 2\right)}$

(16) What language does the following DFA accept?

TYPO, so give everyone credit



A $\{0\}^* \bullet 1 \bullet \{01, 0001\}^*$

B $\{0\}^* \bullet \{100\}^*$

C $\{0,1\}^* \bullet \{1000\}^*$

D $\{0,1\}^* \bullet \{100\}^*$

E $\{0\}^* \bullet \{100, 1000\}^*$

should be $\{0\}^* \bullet \{10, 1000\}^*$

(17) If you have seven colors to choose from, how many different colorings of K_5 can you provide?

A 21

B 35

C 120

D 2520

E 5040

$\binom{10}{5} 5! = \boxed{2520}$

recall: (1) $\neg(A \vee B) \overset{eqv}{\equiv} \neg A \wedge \neg B$

(2) $\neg(A \wedge B) \overset{eqv}{\equiv} \neg A \vee \neg B$

(18) Which is logically equivalent to $\neg((p \wedge q) \vee r)$?

A $(p \vee \neg r) \vee (q \wedge \neg r)$

B $(\neg p \vee r) \wedge (\neg q \wedge r)$

C $(\neg p \wedge r) \vee (q \wedge \neg r)$

D $(\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$

E $(\neg p \wedge \neg r) \vee (\neg q \wedge \neg r)$

$\neg((p \wedge q) \vee r) \underset{\text{by 1}}{\overset{eqv}{\equiv}} \neg(p \wedge q) \wedge \neg r \underset{\text{by 2}}{\overset{eqv}{\equiv}} (\neg p \vee \neg q) \wedge \neg r$

$\underset{\text{distributive property}}{\overset{eqv}{\equiv}} (\neg p \wedge \neg r) \vee (\neg q \wedge \neg r)$

distributive property $\rightarrow (a \vee b) \wedge c = (c \wedge a) \vee (c \wedge b)$

Some students find the phrasing ambiguous

(19) What is the contrapositive of "If a millenial $\underbrace{\text{didn't inherit money or win the lottery}}_{p}$, then $\underbrace{\text{they do not own a home}}_{q}$"?

A If a millenial inherits money and wins the lottery, then they own a home.

B If a millenial owns a home, then they inherited money or won the lottery. $\neg q \rightarrow \neg p$

C If a millenial inherits money or wins the lottery, then they own a home.

D If a millenial owns a home, then they inherited money and won the lottery.

E If a millenial does not own a home, then they did not inherit money or win the lottery.

count either

(20) Which string below is not in the language of the CFG

1: $S \rightarrow 0T1|T1$ $\leftarrow \{0^{2n+1}1^{n+1} \mid n \geq 0\} \cup \{0^{2n}1^{n+1} \mid n \geq 0\}$

2: $T \rightarrow 00T1|\varepsilon$ $\leftarrow \{0^{2n}1^n \mid n \geq 0\}$

A 00000111

B 1

C 00111

D [redacted]

E 0000111

2 You play a game in which you pay one dollar to blindly draw three balls from a box, uniformly at random without replacement. The box contains ten balls; four of these balls are golden. You get back your original one-dollar stake if you draw exactly two golden balls, while you win ten dollars plus your original one-dollar stake if you draw three golden balls. Otherwise, you get nothing. What is your expected loss after playing this game?

Let G be the number of golden balls and L be the loss.

Then

$$L = \begin{cases} 0, & \text{if } G=2 \\ -10, & \text{if } G=3 \\ 1, & \text{otherwise} \end{cases},$$

so

$$\mathbb{E}L = 0\,\mathbb{P}(G=2) - 10\,\mathbb{P}(G=3) + 1\left(1-\mathbb{P}(G=2)-\mathbb{P}(G=3)\right).$$

Note that

$$\mathbb{P}(G=2) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}, \quad \mathbb{P}(G=3) = \frac{\binom{4}{3}}{\binom{10}{3}}.$$

Plugging in gives

$$\boxed{\mathbb{E}L = \frac{1}{3}}$$

3 **Prove that, for any random variable X that takes on values in the finite set $\{a_1, \dots, a_n\}$, there is at least one a_i in this set that satisfies**

$$a_i \geq \mathbb{E}X.$$

Clearly identify the type of proof you used.

Prf

Contradiction. Assume that all the a_i satisfy $a_i < \mathbb{E}X$. Then

$$\mathbb{E}X = \sum_{i=1}^{n} p_i a_i, \quad \text{where } p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1$$

$$< \sum_{i=1}^{n} p_i \mathbb{E}X = \mathbb{E}X.$$

This contradiction (that $\mathbb{E}X < \mathbb{E}X$) shows our assumption is incorrect. Therefore there is some a_i for which it is the case that $a_i \geq \mathbb{E}X$

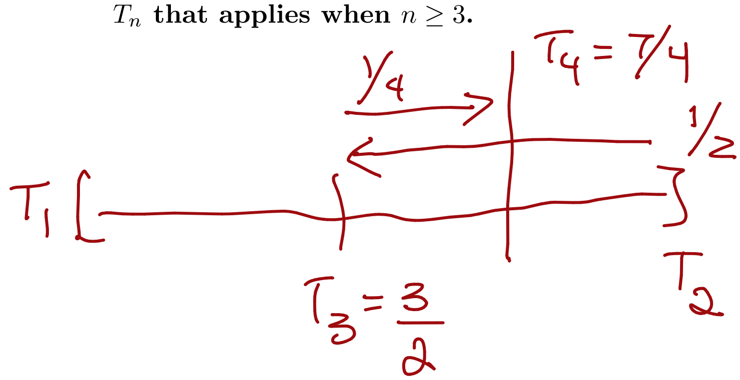
$\blacksquare$

4 **Write pseudocode (at the level of the examples given in class) for solving the problem $\mathcal{L} = \{0^{\bullet n}1^{\bullet(n+2)} \mid n > 0\}$ using a Turing Machine.**

Pseudocode

1. Check format of input.
2. Return to *. Move to next unmarked 0. If there is no unmarked 0 go to step 4. Else mark this 0.
3. Move right to next unmarked 1. If there is none, REJECT. Else mark it and go to step 2.
4. Move right to next unmarked 1. If there is none, REJECT. Else move right. If there is no 1, REJECT. Else move right. If the char is ⊔, ACCEPT

5 Let $T_1 = 1$, $T_2 = 2$ and $T_n = (T_{n-1} + T_{n-2})/2$ when $n \geq 3$. Find a closed form expression for T_n that applies when $n \geq 3$.



$\Rightarrow$ Suggests when $n \geq 3$

$$T_n = 2 - \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \dots + \left(-\frac{1}{2}\right)^{n-2}$$

Guess $T_n = 2 + \sum_{i=1}^{n-2} \left(-\frac{1}{2}\right)^i$

$$= 2 + \sum_{i=0}^{n-2} \left(-\frac{1}{2}\right)^i - 1 = 1 + \frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{1 - \left(-\frac{1}{2}\right)}$$

Up to here is enough for full credit

$\boxed{T_n = 1 + \frac{2}{3}\left(1 + 2\left(-\frac{1}{2}\right)^n\right)}$ when $n \geq 3$

Claim: $T_n = 1 + \frac{2}{3}\left(1 + 2\left(-\frac{1}{2}\right)^n\right)$ when $n \geq 3$

to check validity

Prf

Base case: $n = 3$, $T_n = 3/2$; $n = 4$, $T_n = 7/4$ } match the formula

Strong Induction: Assume ~~that~~ for all $3 \leq k < n$,
$$T_k = 1 + \frac{2}{3}\left(1 + 2\left(-\frac{1}{2}\right)^k\right)$$

Then
$$T_n = \frac{T_{n-1} + T_{n-2}}{2} = 1 + \frac{1}{3}\left(1 + 2\left(-\frac{1}{2}\right)^{n-1}\right) + \frac{1}{3}\left(1 + 2\left(-\frac{1}{2}\right)^{n-2}\right)$$
$$= 1 + \frac{2}{3} + \frac{2}{3}\left[\left(-\frac{1}{2}\right)^{n-1} + \left(-\frac{1}{2}\right)^{n-2}\right] = 1 + \frac{2}{3} + \frac{2}{3}\left[-2\left(-\frac{1}{2}\right)^n + 4\left(-\frac{1}{2}\right)^n\right]$$
$$= 1 + \frac{2}{3}\left(1 + 2\left(-\frac{1}{2}\right)^n\right).$$

6 **Prove that $(3^{77}-1)/2$ is odd.**

Prf Contradiction

Assume it is even, then

$$\frac{3^{77}-1}{2} = 2k \text{ for some } k \in \mathbb{Z}$$

$$\Rightarrow 3^{77}-1 = 4k \Rightarrow 3^{77} \bmod 4 = 1.$$

However, we check that

$$3^{77} \bmod 4 = 3^{2\cdot 38+1} \bmod 4$$

$$= 9^{38}\cdot 3 \bmod 4$$

$$= 3$$

so this contradiction shows that it must be the case that in fact $\frac{3^{77}-1}{2}$ is odd

$\blacksquare$

SCRATCH

SCRATCH