CSCI 6971/4971 Spring 2019 Self-Assessment

Linear Algebra

Concepts: SVD, EVD, QR decompositions; Rank, Nullity; Positivity; Orthonormality; Vector norms; Frobenius norm; Spectral norm

1. What is the tightest upper bound on $|\mathbf{x}^T\mathbf{y}|$ in terms of the Euclidean norms of $\mathbf{x}$ and $\mathbf{y}$?

2. Let matrices $\mathbf{A}$ and $\mathbf{B}$ have the same dimensions; show that the trace of the matrix $\mathbf{A}\mathbf{B}^T$ is the same as the inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$ obtained by stacking the columns of $\mathbf{A}$ and $\mathbf{B}$.

3. Why is it that $\mathbf{A}^T\mathbf{A} + \mathbf{I}$ is invertible for any matrix $\mathbf{A}$?

4. Express $\|\mathbf{A}^{-1}\|_2$ in terms of the singular values of $\mathbf{A}$.

5. If $\mathbf{A} = \mathbf{Q}\mathbf{R}$ is a QR decomposition, give an expression for $\mathbf{A}^T\mathbf{A}$ in terms of $\mathbf{R}$.

6. If $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$ is the full SVD of $\mathbf{A}$, then what are the full SVDs of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$?

7. If $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$ is the full SVD of $\mathbf{A}$, how can you read off the rank and nullity of $\mathbf{A}$ from just $\mathbf{S}$?

8. If $\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = 0$, what can we say about $\mathbf{x}$?

9. If $\mathbf{A}$ is symmetric (not necessarily positive-definite), how are its eigenvalue decomposition and singular value decomposition related?

10. If $\mathbf{U}$ is a matrix with orthonormal columns, argue that $\|\mathbf{U}\mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$.

11. Express the Frobenius norm of $\mathbf{A}$ in terms of the trace of the matrix $\mathbf{A}^T\mathbf{A}$ and argue that it is smaller than the spectral norm of $\mathbf{A}$.

12. If $\mathbf{A}$ and $\mathbf{B}$ are matrices whose columns are respectively $\{\mathbf{a}_i\}_i$ and $\{\mathbf{b}_i\}_i$, show that $\mathbf{A}\mathbf{B}^T = \sum_i a_i b_i^T$.

13. Express $\|\mathbf{a}\mathbf{a}^T\|_F^2$ in terms of the Euclidean length of $\mathbf{a}$.

Probability

Concepts: Independence; Variance; Expectation; Total Probability; Gaussians

1. Let $p(x, y)$ be the joint pdf for two random variables X, Y in $\mathbb{R}^2$; give expressions for $p_1(x)$ and $p_2(y)$, the marginals of X and Y.

2. If X and Y are independent random variables, how is p related to p_1 and p_2?

3. If X and Y are independent, give an expression for the expectation of $f(x)g(y)$, where f and g are arbitrary functions (this expression should not be true in general if X and Y are not independent).

4. What can be said about $\mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\}$ if the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are independent?

5. If X, Y, Z are independent, what can we say about $\mathbb{E}(X+Y+Z)$? If they are not independent?

6. Let $[[\cdot]]$ be the indicator function that returns 1 if the argument is true and 0 otherwise; give a simple expression for $\mathbb{E}([[X \in \mathcal{A}]])$.

7. Use the law of total probability to argue that $\mathbb{P}\{\mathcal{E}_1\} \leq \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} + \mathbb{P}\{\mathcal{E}_2^c\}$.

8. Give a sufficient condition for when the variance of $X + Y$ equals the sum of the variances of X and Y. Provide a proof of your claim.

9. Let $\mathbf{x}$ be a standard multivariate Gaussian in $\mathbb{R}^n$, $\mathbf{O}$ be an orthonormal matrix, and $\mathcal{A}$ be a subset of $\mathbb{R}^n$. Argue that $\mathbb{P}\left\{\mathbf{Ox} \in \mathcal{A}\right\} = \mathbb{P}\left\{\mathbf{x} \in \mathbf{O}^T \mathcal{A}\right\}$, and evaluate the integral for the latter probability to argue that $\mathbf{Ox}$ is also a standard multivariate Gaussian.

10. If $X_1, \ldots, X_n$ are independent identically distributed samples drawn from a distribution with finite variance, what can we say about the distribution of their average $\frac{1}{n} \sum_{i=1}^{n} X_i$?