Guarantees on the performance of the CX algorithm when uniform column sampling is used

<u>Setup</u>: $A \in \mathbb{R}^{n \times d}$ is an arbitrary rectangular matrix (so d may be larger than n)

$S \in \mathbb{R}^{d \times c}$ is our random column sampling matrix:

$$S = \sqrt{\frac{d}{c}}\left[s_1, \dots, s_c\right]$$

with i.i.d. columns, where

$$\mathbb{P}(s_i = e_j) = \frac{1}{d} \quad \text{for } j = 1, \dots, d$$

<u>Alg</u>: Take $C = AS$

Solve $X_{opt} = \underset{X \in \mathbb{R}^{c \times d}}{\operatorname{argmin}} \|A - CX\|_F^2 = C^+ A$

Let our low-rank approximation be CX_{opt}

<u>Guarantee</u>: if $c \geq \frac{2592 d}{\varepsilon^2} \mu(A_k^T) \ln\left(\frac{k}{\delta}\right)$, then

$$\|A - CX_{opt}\|_F^2 \leq (1+\varepsilon)\|A - A_k\|_F^2$$

with probability at least $1 - \delta - \frac{\varepsilon c}{d \mu(A_k^T)}$

Recall that the coherence $\mu(B)$ of a matrix is defined as the largest squared Euclidean norm of the rows of an orthonormal basis for the range of B.
So if $B = QR$ or $B = U\Sigma V^T$ (compact SVD), with
$Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$, $U = \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$, then

$$\mu(B) = \max_i \|q_i\|_2^2 = \max_i \|u_i\|_2^2 = \mu(Q) = \mu(U)$$

Note $\mu(B)$ is independent of the specific choice of onb

Hence if $A_k = U_1 \Sigma_1 V_1^T$, then

$$\mu(A_k^T) = \mu(V_1) = \max_i \|e_i^T V_1\|_2^2$$

The <u>leverage scores</u> $\|e_i^T V_1\|_2^2$ measure how important each of the d columns of A are in determining the best rank-k approximation to A, A_k.
The coherence is high iff there are some columns that must be used to get a good approximation to A_k.

In that case, a coupon collector argument says we will need to sample $O(d \log d)$ columns uniformly with replacement to see the important columns.

The guarantee from before gives this kind of estimate. Only when the coherence is small is uniform sampling useful.

Results we will use to get the guarantee

- Recall that $\langle A, B\rangle = \mathrm{Tr}(AB^T)$ for two matrices, and $\|A\|_F^2 = \langle A, A\rangle$; in particular, we will use the corollary that
$$\|A - B\|_F^2 = \langle A-B, A-B\rangle$$
$$= \|A\|_F^2 - 2\langle A, B\rangle + \|B\|_F^2,$$
so if $AB^T = 0$, then
$$\|A - B\|_F^2 = \|A\|_F^2 + \|B\|_F^2$$

- If X is a matrix, then XX^+ is the orthogonal projector onto the range of X. This implies that if X has full row rank, then XX^+ has full-rank so is invertible. The only invertible projector is the identity.

- If X has full row-rank, then $X^+ = X^T(XX^T)^{-1}$ (you can show this with the SVD)

- Weyl's Perturbation Thm: if A and B are symmetric matrices, then $|\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|_2$ for all eigenvalues

- For any matrix $A \succeq 0$, all its eigenvalues are ≥ 0, and the SVD is the same as the EVD; in particular,

$$\lambda_i(A) = \sigma_i(A) \text{ for all } i$$

- For any matrix X, $\|X^+\|_2 = \sigma_{\min}(X)^{-1}$ (this is clear from the SVD)

- A new matrix multiplication result: if $A \in \mathbb{R}^{n \times d}$, $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_d^T \end{bmatrix} \in \mathbb{R}^{d \times k}$, and $S \in \mathbb{R}^{d \times c}$ is the usual uniform column selection matrix, then when $c \geq \frac{Md}{\varepsilon} \max_i \|b_i\|_2^2$,

$$\mathbb{P}\left(\|ASS^TB - AB\|_F^2 \geq \varepsilon \|A\|_F^2\right) \leq \frac{1}{M}$$

(proof given later)

- For any two matrices, $\|AB\|_F \leq \begin{cases} \|A\|_2 \|B\|_F \\ \|A\|_F \|B\|_2 \end{cases}$

Proof of CX performance guarantee

Idea:

$$\|A - CX_{opt}\|_F^2 \leq \min_X \|A - CX\|_F^2$$

so it suffices to demonstrate the existence of an $\tilde{X}$ for which we know

$$\|A - C\tilde{X}\|_F^2 \leq (1+\varepsilon)\|A - A_k\|_F^2$$

w.h.p.

This would then imply that

$$\|A - CX_{opt}\|_F^2 \leq (1+\varepsilon)\|A - A_k\|_F^2$$

The key intuition for finding an $\tilde{X}$ is that

$$C = A_k S + A_{\rho-k} S = U_1 \Sigma_1 V_1^T S + U_2 \Sigma_2 V_2^T S$$

If S has a small angle against V_1 and a large angle against V_2, then

$$C \approx U_1 \Sigma_1 V_1^T S \Rightarrow CX_{opt} = CC^+ A = P_C A \approx P_{U_1} A = A_k$$

as we want.

In particular, for CX_{opt} to be almost as good as A_k, we require C to contain all the range of U_1.

This happens if $rk(V_1^T S) = k$, b/c then

$$A_k S = U_1 \Sigma_1 V_1^T S \quad \text{has full rank } k$$

which means it contains the entire range of U_1

Further, the form of $A_k S$ suggests taking $\tilde{X} = (V_1^T S)^+ V_1^T$ and assuming $\tilde{X}$ has rank k, because with this choice

$$C\tilde{X} = A_k S\tilde{X} + A_{\rho-k} S\tilde{X}$$
$$= U_1\Sigma_1(V_1^TS)(V_1^TS)^+V_1^T + U_2\Sigma_2V_2^TS(V_1^TS)^+V_1^T$$
$$= U_1\Sigma_1V_1^T + U_2\Sigma_2V_2^TS(V_1^TS)^+V_1^T = A_k + \mathcal{E}$$

where we used the fact that $(V_1^TS)(V_1^TS)^+$ is a $k \times k$ full-rank projection, so must be the identity, and the intuition that

$$V_2^TS(V_1^TS)^+ \approx V_2^TSSV_1 \approx V_2^TV_1 \approx 0,$$

to be justified later.

Using this $\tilde{X}$, we have

$$\|A - CX_{opt}\|_F^2 \leq \|A - C\tilde{X}\|_F^2$$
$$= \|A_k + A_{\rho-k} - (A_k + U_2\Sigma_2V_2^TS(V_1^TS)^+V_1^T)\|_F^2$$
$$= \|A_{\rho-k} - A_{\rho-k}S(V_1^TS)^+V_1^T\|_F^2$$

and note that

$$A_{\rho-k}S(V_1^TS)^+V_1^TA_{\rho-k}^T = U_2\Sigma_2V_2^TS(V_1^TS)^+V_1^TV_2\Sigma_2U_2^T$$
$$= 0$$

so

$$= \|A_{\rho-k}\|_2^2 + \|A_{\rho-k}S(V_1^TS)^+V_1^T\|_F^2$$

Now we use our assumption that $V_1^T S$ has full row rank k to see that

$$(V_1^T S)^+ = S^T V_1 (V_1^T S S^T V_1)^{-1}$$

so

$$\|A - CX_{opt}\|_F^2 \le \|A - C\tilde{X}\|_F^2$$
$$= \|A_{\rho-k}\|_F^2 + \|A_{\rho-k} S (V_1^T S)^+ V_1^T\|_F^2$$
$$\le \|A_{\rho-k}\|_F^2 + \|A_{\rho-k} S S^T V_1 (V_1^T S S^T V_1)^{-1} V_1^T\|_F^2$$
$$\le \|A_{\rho-k}\|_F^2 + \|(V_1^T S S^T V_1)^{-1} V_1^T\|_2^2 \|A_{\rho-k} S S^T V_1\|_F^2$$
$$\le \|A_{\rho-k}\|_F^2 + \|(V_1^T S S^T V_1)^{-1}\|_2^2 \|A_{\rho-k} S S^T V_1\|_F^2$$
$$= \|A_{\rho-k}\|_F^2 + \underbrace{\lambda_k (V_1^T S S^T V_1)^{-2}}_{=: T_1} \underbrace{\|A_{\rho-k} S S^T V_1\|_F^2}_{=: T_2}$$

Now we show that $T_1 \approx 1$ and $T_2 \approx \varepsilon \|A_{\rho-k}\|_F^2$.

By Weyl's Perturbation Theorem,

$$|\lambda_k (V_1^T S S^T V_1) - 1| \le \|V_1^T S S^T V_1 - I\|_2$$

and from our earlier application of the Matrix Bernstein to subspace embeddings via row sampling, we know that if

$$c \ge \frac{2592}{\varepsilon^2} d \mu(V_1) \ln\left(\frac{k}{2\delta}\right)$$

then this quantity is smaller than $\frac{\varepsilon}{18}$. Note that this is exactly the requirement we imposed on c

It follows that

$$\lambda_k(V_1^T S S^T V_1) \geq 1 - \frac{\varepsilon}{18} \Rightarrow$$

$$T_1 = \lambda_k(V_1^T S S^T V_1)^{-2} \leq \left(1 - \frac{\varepsilon}{18}\right)^{-2} \leq (1 + \varepsilon/6)^2 \leq 1 + \frac{\varepsilon}{2}$$

Now consider T_2. Apply our new matrix multiplication result with $A = A_{\rho-k}$ and $B = V_1$ to see that

$$\| A_{\rho-k} S S^T V_1 - A_{\rho-k} V_1 \|_F^2 \leq \frac{\varepsilon}{2} \| A_{\rho-k} \|_F^2$$

with probability at least $1 - \frac{1}{m}$

when $c \geq \frac{2md}{\varepsilon} \mu(V_1)$

Notice that:

1) $A_{\rho-k} V_1 = U_2 \Sigma_2 V_2^T V_1 = 0$, so this implies $T_2 \leq \frac{\varepsilon}{2} \| A_{\rho-k} \|_F^2$

2) We can define $m = \frac{\varepsilon c}{2 d \mu(V_1)}$

It follows that

$$\begin{aligned} \| A - C X_{opt} \|_F^2 &\leq \| A_{\rho-k} \|_F^2 + T_1 T_2 \\ &\leq \| A_{\rho-k} \|_F^2 + \left(1 + \frac{\varepsilon}{2}\right) \frac{\varepsilon}{2} \| A_{\rho-k} \|_F^2 \\ &\leq (1 + \varepsilon) \| A - A_k \|_F^2 \end{aligned}$$

w.p. at least $1 - \delta - \frac{\varepsilon c}{2 d \mu(V_1)}$

$\blacksquare$

Proof of the Matrix Multiplication Result

Lemma

Let $A \in \mathbb{R}^{n \times d}$, $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_d^T \end{bmatrix} \in \mathbb{R}^{d \times k}$, and let

$S = \sqrt{\frac{d}{c}}\,[s_1, \ldots, s_c]$ be a random column selection matrix with s_i i.i.d and $\mathbb{P}(s_i = e_j) = \frac{1}{d}$ for $j = 1, \ldots, d$.

Then

$$\mathbb{E}\,\| ASS^TB - AB \|_F^2 \le \frac{d}{c} \max_i \|b_i\|_2^2 \, \|A\|_F^2$$

Prf

$$\mathbb{E}\,\| ASS^TB - AB \|_F^2 = \mathbb{E}\left[\| ASS^TB \|_F^2 - 2\langle AB, ASS^TB \rangle + \|AB\|_F^2 \right]$$

$$= \mathbb{E}\,\| ASS^TB \|_F^2 - 2\|AB\|_F^2 + \|AB\|_F^2$$

$$= \mathbb{E}\,\| ASS^TB \|_F^2 - \|AB\|_F^2$$

Now let g_i, $i = 1, \ldots, d$ denote the number of times that column i is sampled by S, so

$$ASS^TB = \frac{d}{c} \sum_{i=1}^{c} g_i a_i b_i^T$$

Note that $g_i \sim \mathrm{Bin}\left(\frac{1}{d}, c\right)$ and the g_i are not independent.

Then

$$\mathbb{E}\,\| ASS^TB \|_F^2 = \frac{d^2}{c^2}\,\mathbb{E}\left\langle \sum_{i=1}^{c} g_i a_i b_i^T, \sum_{j=1}^{c} g_j a_j b_j^T \right\rangle$$

$$= \frac{d^2}{c^2} \mathbb{E}\left[\sum_{i=1}^{c} \sum_{j=1}^{c} g_i g_j \langle a_i b_i^T, a_j b_j^T \rangle \right]$$

$$= \frac{d^2}{c^2} \sum_{i=1}^{c} \sum_{j=1}^{c} \mathbb{E}[g_i g_j] \, Tr(a_i b_i^T b_j a_j^T)$$

$$= \frac{d^2}{c^2} \sum_{i=1}^{c} \sum_{j=1}^{c} \mathbb{E}[g_i g_j] \langle b_i, b_j \rangle \langle a_i, a_j \rangle$$

To evaluate this, note that if $i = j$, then we can write

$$\mathbb{E}(g_i^2) = \mathbb{E}\left(\sum_{l=1}^{c} b_i \right)^2$$

where b_i are i.i.d. $Bern\left(\frac{1}{d}\right)$ r.v.s

$$= \mathbb{E}\left(\sum_{l=1}^{c} \sum_{p=1}^{c} b_i b_p \right) = \sum_{l=1}^{c} \mathbb{E}(b_i^2) + \sum_{l=1}^{c} \sum_{\substack{p=1 \\ p \neq l}}^{c} \mathbb{E}(b_i) \mathbb{E}(b_j)$$

$$= \frac{c}{d} + \frac{c(c-1)}{d^2}$$

When $i \neq j$, we observe that $\sum_{i=1}^{d} g_i = c$ and

$$\mathbb{E}\left(\sum_{i=1}^{d} g_i \right)^2 = c^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \mathbb{E}(g_i g_j)$$

$$= \sum_{i=1}^{d} \mathbb{E}(g_i)^2 + d(d-1) \mathbb{E}[g_1 g_2]$$

$$= c + \frac{c(c-1)}{d} + d(d-1) \mathbb{E}[g_1 g_2]$$

$$\Rightarrow \mathbb{E}[g_1 g_2] = \frac{c(c-1)}{d^2} = \mathbb{E}[g_i g_j] \text{ for } i \neq j$$

so

$$\mathbb{E}\|ASS^TB - AB\|_F^2 = \mathbb{E}\|ASS^TB\|_F^2 - \|AB\|_F^2$$

$$= \frac{d^2}{c^2}\sum_{i=1}^{d}\sum_{j=1}^{d}\mathbb{E}[g_ig_j]\langle a_i, a_j\rangle\langle b_i, b_j\rangle - \|AB\|_F^2$$

and we've shown

$$\mathbb{E}[g_ig_j] = \frac{c(c-1)}{d^2} + \frac{c}{d}\mathbb{1}\{i=j\}$$

so

$$= \frac{d^2}{c^2}\left[\sum_{i=1}^{d}\frac{c}{d}\|a_i\|_2^2\|b_i\|_2^2 + \frac{c(c-1)}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d}\langle a_i, a_j\rangle\langle b_i, b_j\rangle\right]$$

$$- \|AB\|_F^2$$

$$= \frac{d}{c}\sum_{i=1}^{d}\|a_i\|_2^2\|b_i\|_2^2 + \frac{c-1}{c}\|AB\|_F^2 - \|AB\|_F^2$$

$$\leq \frac{d}{c}\max_i\|b_i\|_2^2\,\|A\|_F^2$$

$\blacksquare$

Matrix Mult Lemma

Let A, B, S be as before, then

$$\mathbb{P}\left(\|ASS^TB - AB\|_F^2 \geq \varepsilon \|A\|_F^2\right) \leq \frac{1}{m}$$

when $c \geq \frac{m}{\varepsilon} d \max_i \|b_i\|_2^2$

Prf

$$\mathbb{P}\left(\|ASS^TB - AB\|_F^2 \geq \varepsilon \|A\|_F^2\right) \leq \frac{\mathbb{E}\|ASS^TB - AB\|_F^2}{\varepsilon \|A\|_F^2}$$

$$\leq \frac{d \max_i \|b_i\|_2^2 \, \|A\|_F^2}{c\,\varepsilon \|A\|_F^2}$$

$$\leq \frac{1}{m}$$

$\blacksquare$