

CSCI 6220/4030: Homework 5

Assigned Thursday November 7 2019. Due at beginning of class Thursday November 21 2019.

Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours. Provide mathematically convincing arguments for the following problems. Ask me if you are unclear whether your arguments are acceptable.

1. Consider two computers, each containing n bit strings of length n . It can be shown that any deterministic algorithm for determining whether these sets have a non-empty intersection requires $O(n^2)$ bits to be communicated between the computers.
 - (a) Design a one-sided Monte Carlo algorithm for answering this problem that uses one round of communication, communicates $O(n \log n)$ bits, always succeeds when the sets intersect and fails with probability less than $n^{-\varepsilon}$ when the sets do not intersect. Here $\varepsilon > 0$ is a constant used as input to the algorithm. Give an algorithm listing and an analysis of the failure probability.
 - (b) Design a Las Vegas algorithm for answering this problem that uses two rounds of communication and communicates $O(n \log n)$ bits in expectation *when the sets do not intersect*. Given an algorithm listing, a proof that the algorithm is always correct, and an analysis of the expected number of bits transmitted.
2. Recall in class that we said a continuously differentiable function f is convex iff $\nabla^2 f(x) \succeq 0$ at every point in the domain. That is, the eigenvalues of the Hessian of f are everywhere non-negative; geometrically, this means that f has non-negative curvature at every point.
 - (a) Prove that $\exp(\cdot)$ is a convex function with domain \mathbb{R} .
 - (b) Use Jensen's inequality to show that a convex function f has the following property: for any $\alpha \in [0, 1]$ and points x and y in the domain of f ,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

Geometrically, this means that the graph of f lies below its secant lines. Since this property can be satisfied by non-differentiable functions and clearly generalizes the idea of having positive curvature everywhere, this is usually taken as the definition of a convex function¹.

- (c) Similarly, show that if f is convex and $\sum_{i=1}^n \alpha_i = 1$, where all the α_i are nonnegative, then for any points x_1, \dots, x_n in the domain of f ,

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i).$$

- (d) Use the fact that $\exp(\cdot)$ is a convex function with domain \mathbb{R} to prove that if $\sum_{i=1}^n \alpha_i = 1$ and all the α_i are nonnegative, then for any positive numbers x_1, \dots, x_n ,

$$\prod_{i=1}^n x_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i x_i.$$

This result generalizes the usual statement of the arithmetic-geometric mean inequality:

$$(x_1 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}.$$

¹One can show that any continuously differentiable function that lies below its secants must satisfy $\nabla^2 f \succeq 0$, so the definitions are consistent.

3. [required only for CSCI6220] Let p and q be positive numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Recall the p -norm of a vector \mathbf{x} , denoted $\|\mathbf{x}\|_p$, satisfies

$$\|\mathbf{x}\|_p^p = \sum_{i=1}^n |x_i|^p.$$

- (a) Let a, b be strictly positive numbers and select positive numbers for which $\frac{1}{p} + \frac{1}{q} = 1$. Use Jensen's inequality to argue that

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

- (b) Prove that if all the entries of \mathbf{x} and \mathbf{y} are strictly positive,

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \frac{1}{p} \|\mathbf{x}\|_p^p + \frac{1}{q} \|\mathbf{y}\|_q^q,$$

and therefore, for arbitrary vectors \mathbf{x} and \mathbf{y} ,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \frac{1}{p} \|\mathbf{x}\|_p^p + \frac{1}{q} \|\mathbf{y}\|_q^q.$$

- (c) Now argue that this implies the following (very useful) inequality: for any vectors \mathbf{x} and \mathbf{y} ,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

Observe that if $p = q = 2$, then this reduces to the usual Cauchy-Schwarz inequality. If $p = 1$ and $q = \infty$, then² we recover the obvious fact that

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \left(\sum_{i=1}^n |x_i| \right) \cdot \max_{i=1}^n |y_i|.$$

- (d) The inner product between two random variables X and Y is defined as

$$\langle X, Y \rangle = \mathbb{E}(XY) = \sum_{\omega \in \Omega} X(\omega)Y(\omega)p(\omega),$$

and the p -norm of a random variable X , denoted $\|X\|_p$, satisfies

$$\|X\|_p^p = \mathbb{E}(|X|^p).$$

Argue that we still have

$$|\langle X, Y \rangle| \leq \|X\|_p \|Y\|_q$$

when p and q are positive numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. When $p = q = 2$ and $\mathbb{E}X = \mathbb{E}Y = 0$, this is the statement

$$\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X)\text{Var}(Y)}.$$

²This inequality still holds, by a limiting argument. But this argument isn't needed since we are working with finite-dimensional vectors. It is necessary once we make this claim for the version of the inequality that applies to *continuous* random variables.