CSCI 6220/4030: Homework 3

Assigned Thursday October 17 2018. Due at beginning of class Monday October 28 2018.

Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours. Provide mathematically convincing arguments for the following problems. Ask me if you are unclear whether your arguments are acceptable.

1. Let X be a Binomial(n, p) random variable.
   (i) Determine the MGF of X.
   (ii) Use the MGF of X to determine a subgaussian tail bound for X.

2. Let X be a Poisson(λ) random variable and $\varepsilon > 0$.
   (i) Compute the MGF of X.
   (ii) Compute the mean, μ, and variance, $\mathrm{Var}(X)$, of X.
   (iii) Use Chebyshev's inequality to derive a bound on the probability that $X > (1+\varepsilon)\mathbb{E}X$.
   (iv) Use the Chernoff technique to argue that
   $$\mathbb{P}(X \geq (1+\varepsilon)\mathbb{E}X) \leq \mathrm{e}^{-h(\varepsilon)\mathrm{Var}(X)},$$
   where $h(\varepsilon) = (1+\varepsilon)\ln(1+\varepsilon) - \varepsilon$.
   (v) Compare the usefulness of the two bounds.

3. Consider a collection $X_1, \ldots, X_n$ of n independent geometrically distributed random variables with mean 2. Let $X = \sum_{i=1}^n X_i$ and $\delta > 0$.
   (i) Derive a bound on $\mathbb{P}(X \geq (1+\delta)(2n))$ by relating this probability to the behavior of a sum of $(1+\delta)2n$ Bernoulli random variables and applying a Chernoff bound.
   (ii) Directly derive a Chernoff bound on $\mathbb{P}(X \geq (1+\delta)(2n))$ using the moment generating function for geometric random variables. The form of the bound should be simple.
   (iii) Which bound is better in your opinion, and why?

4. [**required only for CSCSI6220**] Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the ith bin is empty and 0 otherwise. Let $X = \sum_i X_i$. Note that X is *not* the sum of independent random variables, so we cannot use a Chernoff bound directly.

   Instead, we will show that the MGF of X is smaller than the MGF of a sum of independent r.v.s and obtain a Chernoff bound in terms of the latter. To do so, let Y_i for $i = 1, \ldots, n$ be independent Bernoulli random variables that are 1 with probability $p = (1 - 1/n)^n$. Let $Y = \sum_{i=1}^n Y_i$.
   (i) What is the probability that $X_i = 1$?
   (ii) Give an intuitive argument for why we should expect Y to be greater than X.
   (iii) Show that $\mathbb{E}[X_1 X_2 \cdots X_k] \leq \mathbb{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.
   (iv) Show that $M_X(\lambda) = \mathbb{E}[\mathrm{e}^{\lambda X}] \leq \mathbb{E}[\mathrm{e}^{\lambda Y}] = M_Y(\lambda)$ for all $\lambda \geq 0$. (Hint: use the expansion for e^x and compare $\mathbb{E}[X^k]$ to $\mathbb{E}[Y^k]$.)
   (v) Derive a Chernoff bound for $\mathbb{P}(X \geq (1+\delta)\mathbb{E}[X])$.