

CSCI 6220/4030: Homework 1

Assigned Thursday September 5 2019. Due at beginning of class Monday September 23 2019.

Provide well-written, clear, and convincing arguments for the following problems. Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours.

1. Calculate the expected sizes of the sets $S_{<x}$ and $S_{>x}$ in the first step of the algorithm when the randomized quicksort algorithm is used to sort n elements.

HINT: Express the sizes of the sets as the sum of indicator random variables.

2. Grandma brings r pieces of candy to the family reunion, where n kids are in attendance. Luckily for them, $r \geq n$. Grandma has a bad memory, so she doesn't remember who already has gotten a piece of candy, and gives out each piece of candy to a uniformly randomly chosen kid. What is the probability that every kid gets a piece of candy?

HINT: Use the Inclusion-Exclusion Principle.

3. Suppose we flip a coin n times to obtain a sequence of flips X_1, \dots, X_n . A streak of flips is a consecutive subsequence of flips that are all the same. For example, if X_3, X_4 , and X_5 are all heads, there is a streak of length 3 starting at the third flip. (NB: if X_6 is also heads, then there is also a streak of length 4 starting at the third flip.) Let n be a power of 2. Show that the expected number of streaks of length $\log_2 n + 1$ is $1 - o(1)$.

HINT: Express the number of streaks as a sum of indicator random variables.

4. Modify the reservoir algorithm so that instead of sampling an element uniformly from the stream, it samples a set of two elements uniformly from the stream. Namely, give an algorithm that maintains counters X_1, X_2 which have the property that after n elements a_1, \dots, a_n are seen,

$$\mathbb{P}[\{X_1, X_2\} = \{a_i, a_j\}] = \frac{1}{\binom{n}{2}} \quad \text{for any subset } \{i, j\} \text{ of } \{1, \dots, n\}.$$

HINT: Maintain two counters, flip a fair coin to determine which to overwrite whenever a new element is seen, and assign element a_i to the chosen counter with a probability *different than* $1/i$ — determine a good choice.

5. Here is another randomized min-cut algorithm: Given a connected unweighted simple graph $G = (V, E)$, assign the edges weights that are chosen independently and uniformly at random in $[0, 1]$ to obtain the weighted graph $G' = (V, E, W)$, run Kruskal's algorithm to compute a minimum weight spanning tree T for G' , identify the edge of T that has largest weight, and return the cut generated by removing that edge¹. Argue that this algorithm returns a mincut on G with probability at least $\binom{n}{2}^{-1}$ in time $O(|E| \log |V|)$.

HINT: The joining of subtrees in this randomized Kruskal's algorithm is analogous to the contraction of edges in Karger's min-cut algorithm.

6. Let $\text{Random}(i, n)$ return a sample of a uniform random choice of an integer in the interval $[i, n]$. Show that, given an array $A[1], \dots, A[n]$, the following algorithm returns an A whose entries have been uniformly randomly permuted:

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1: for  $i \leftarrow 1 \dots n$  do
2:    $r \leftarrow \text{Random}(i, n)$ 
3:   Exchange  $A[i]$  and  $A[r]$ 
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¹This cut is the partition of the vertices given by the two subtrees.

That is, show that if i_1, \dots, i_n is any permutation of $1, \dots, n$,

$$\mathbb{P}[A[1] \leftarrow A[i_1] \text{ and } \dots \text{ and } A[n] \leftarrow A[i_n]] = \frac{1}{n!}.$$

HINT: Analyze the output of this algorithm sequentially by calculating the probability of future assignments conditionally on the assignments that have already been made.

7. (CSCI6220 students) The n passengers on a flight with n seats have been told their seat numbers. They get on the plane one by one. The first person sits in the wrong seat. Subsequent passengers sit in their assigned seats when possible, or otherwise in a randomly chosen empty seat.

Argue that $\Omega(n)$ people end up in their assigned seats with probability greater than $\frac{1}{2}$. What is the probability that the last passenger finds their seat free?

HINT: What happens to the first $k - 1$ passengers if passenger 1 sits in seat k ? How is the behavior of the remaining passengers related to the behavior of a version of the same problem with less passengers?