Condorcet Winner Probabilities - A Statistical Perspective

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Abstract

A Condorcet voting scheme chooses a winning candidate as one who defeats all others in pairwise majority rule. We provide a review which includes the rigorous mathematical treatment for calculating the limiting probability of a Condorcet winner for any number of candidates and value of n odd or even and with arbitrary rank order probabilities, when the voters are independent. We provide a compact and complete Table for the limiting probability of a Condorcet winner with three candidates and arbitrary rank order probabilities. We present a simple proof of a result of May to show the limiting probability of a Condorcet winner tends to zero as the number of candidates tends to infinity. We show for the first time that the limiting probability of a Condorcet winner for any given number of candidates m is monotone decreasing in m for the equally likely case. This, in turn, settles the conjectures of Kelly and Buckley and Westen for the case $n \to \infty$. We prove the validity of Gillett's conjecture on the minimum value of the probability of a Condorcet winner for $m = 3$ and any n. We generalize this result for any m and n and obtain the minimum solution and the minimum probability of a Condorcet winner.

1 Introduction

Different voting procedures (see for example [22] Page 147) for electing a leader or deciding a candidate or a judgment have been proposed in the literature. In all voting procedures, we assume that the voter rank orders the candidate or decision from highest to lowest. One of the most used schemes is a majority rule, in which a candidate or decision is ranked first by more than half the number of voters. An example of the majority rule is the U.S. Supreme Court decisions. Another selection strategy is the plurality rule in which a candidate is ranked first by the largest number of voters. An example of the plurality rule is the U.S. presidential election. In the Condorcet scheme, a candidate is chosen who defeats all others in pairwise majority rule. An example of the Condorcet scheme is the selection of job candidates or proposals to be funded in academic departments and by funding agencies or for determining winners in tournaments.

While the Condorcet scheme is a transitive in individual voter's choices, there are many instances in which a majority winner may not exist [23][19]. For example, let there be 3 voters (V_1, V_2, V_3) and 3 candidates (C_1, C_2, C_3). V_1 rank orders $C_1C_2C_3$, V_2 rank orders $C_2C_3C_1$ and finally V_3 rank orders $C_3C_1C_2$. In this voting example, C_1 is preferred over C_2 two out of three times. Similarly C_2 is preferred over C_3 and C_3 is preferred over C_1 two out of three times. Hence there is no one with majority preferences over all pairs. So according to the Condorcet scheme, there is no winner in this case.

In past research work, it has been assumed that a probability distribution is given to all the rank orders. It is also assumed in most studies that all voters vote independently. Previous researchers have calculated the probability of a Condorcet winner using analytical and simulation models [17] by making simplifying assumptions on the probability distributions. Asymptotic expressions for the limiting probability of a Condorcet winner (i.e., when the number of voters $\to \infty$) have been obtained [14] [12] [9] [23] [19]. A survey of previous research in this area may be found in [10].

The focus of the present paper is on the limiting probability of a Condorcet winner when the number of voters, n, tend to infinity for a given number m of candidates and arbitrary probability distribution on the $m!$ rankings. We present a review and some new results in this part of the study. While researchers in this area, notably for example, Niemi and Weisberg [23], Garman and Kamien [9] have noted the relationship of the limit with certain multivariate normal probabilities, there doesn't exist a statistically rigorous proof of this result in order to justify the evaluation of the limit for various cases. Further it is often assumed that the number of candidates is odd, see Niemi and Weisberg [23]. We close this gap with this paper by providing a rigorous statistical proof for any number of candidates, the number of voters both odd and even, and arbitrary probability distributions, usually referred to as culture probabilities. In section 2, we show further that for any number of candidates and with arbitrary probability distributions on the preference ranking, the limiting probability depends on the calculation of positive orthant probabilities of appropriate multinormal distributions. We also provide a compact and complete Table covering all possible cases for computing the limiting probability of a Condorcet winner when the number of candidates is 3 and with arbitrary probability distribution among the rank orders. We show that the limiting probability as $n \to \infty$ for any number of candidates m is monotone decreasing in m which partially validates Kelly [18] and Buckley and Westen [4] conjectures. We also present a simple proof of May's result [19] that the limiting probability of a Condorcet winner for equally likely case tends to zero as the number of candidates gets larger. We also prove Gillett's [15] conjecture on the minimum probability of a Condorcet number for 3 candidates and any number of voters. We extend the result to any m and n and obtain the minimum value and solution. In Section 5, we treat the case of $m = 4$ candidates with arbitrary probabilities p_i and show that it is possible to have exact expressions for the limiting Condorcet winner probability for all possible scenarios.

2 Condorcet Winner: Statistical Formulation and Results for General Case

In this section, we present a statistical derivation of the limiting probability of a Condorcet winner when the voters are independent and the number of voters is large. Let

$$
\begin{aligned}
m &= \text{Number of Candidates} \\
n &= \text{Number of Voters}
\end{aligned}
$$

We have $K = m!$ preference rankings of m candidates. A voter will choose one of these rankings. We assume that the voters act independently. Let p_i be the probability that a voter prefers the rank order i, for $i = 1, 2, \cdots, K$. Further, we know that $p_i \geq 0$ and $\sum_{i=1}^{K} p_i = 1$. Let N_i be the number of voters voting for the ith preference ranking, for $i = 1, 2, \cdots, K$. We therefore have $\sum_{i=1}^{K} N_i = n$. Now, it is well known that $(N_1, N_2, \cdots, N_K)$ has a multinomial distribution.

$$
P(N_i = n_i, i = 1, \cdots, K) = \frac{n!}{n_1! n_2! \cdots n_K!} \prod_{i=1}^{K} p_i^{n_i} \tag{1}
$$

where the n_i's are nonnegative integers with $\sum_{i=1}^{K} n_i = n$.

It is known that (see, for example,[5] Page 318), for $i = 1, \cdots, K$, $j = 1, \cdots, K$ and $i \neq j$.

$$
\begin{aligned}
E[N_i] &= np_i, \\
Var[N_i] &= np_i(1 - p_i), \\
Covariance(N_i, N_j) &= -np_i p_j, \\
Correlation(N_i, N_j) &= -\sqrt{\frac{p_i p_j}{(1 - p_i)(1 - p_j)}}
\end{aligned}
$$

Denote the m candidates by $C_1, C_2, \cdots, C_m$. Write $C_i P C_j$, $i \neq j$ if C_i is preferred to C_j by a voter. Define $C_i M C_j$, $i \neq j$ if C_i beats C_j by majority rule. i.e., the total number of voters preferring C_i to C_j is more than the total number of voters preferring C_j to C_i. This fact can be mathematically expressed as

$$a_{1,(i,j)}N_1 + a_{2,(i,j)}N_2 + \cdots + a_{K,(i,j)}N_K \geq 1 \tag{2}$$

where the $a_{l,(i,j)}$'s are $\pm$ 1. $a_{l,(i,j)} = 1$, if in the lth preference ranking voted by the N_l voters, $C_i P C_j$. Similarly $a_{l,(i,j)} = -1$ if $C_j P C_i$ is the preference ranking voted by N_l voters. Clearly $\frac{K}{2}$ of the $a_{l,(i,j)}$'s are 1 and $\frac{K}{2}$ of the $a_{l,(i,j)}$'s are equal to -1.

Equation (2) implies that the difference between the number of voters preferring C_i to C_j and the number of voters preferring C_j to C_i is at least 1. We define, for example, C_i to be the Condorcet winner if and only if $C_i M C_j$ for each $j = 1, 2, \cdots, m$ and $j \neq i$. For example, the probability that C_1 is the Condorcet winner is the probability of the joint event

$$\begin{aligned}
a_{1,(1,2)}N_1 + a_{2,(1,2)}N_2 + \cdots + a_{K,(1,2)}N_K &\geq 1 \\
a_{1,(1,3)}N_1 + a_{2,(1,3)}N_2 + \cdots + a_{K,(1,3)}N_K &\geq 1 \\
\cdots &\geq 1 \\
\cdots &\geq 1 \\
a_{1,(1,m)}N_1 + a_{2,(1,m)}N_2 + \cdots + a_{K,(1,m)}N_K &\geq 1
\end{aligned} \tag{3}$$

Condition (3) lead to what is defined as 'strong' winner. Replacing all the 1's by 0's lead to the definition of a 'weak' winner. For small values n and m , this probability can be computed using the multinomial distribution (1). We are interested in finding

$\lim_{n\to\infty} P(\text{One of the } C_i \text{ is a Condorcet winner})$ for a given value of m.

Several researchers have attempted to find this limit or an approximation to it. Gilbaud [14] gave an expression of this limit for $m = 3$, when all the p_i's are equal (Impartial Culture or IC model), though he did not explain the method. Garmen and Kamien [9] derived the expression for the limiting Condorcet winner probability for $m = 3$ and $m = 4$ for the Impartial Culture model without giving details. Jones et al [17], Bell [2] and Niemi et al [23] obtained approximations to it by extensive simulations or quadrature methods attributed to Ruben [25] for the Impartial Culture (IC) model i.e., when all the preference rankings are equal.

Niemi and Weisberg [23] and Garman and Kamien [9] were the first to relate the limit or an approximation to it with the calculation of certain multivariate normal probabilities. Niemi et al [23] assume n to be an odd number and Garman et al [9] indicate the relation in a footnote. However, they do not present a rigorous statistical proof that the limit is exactly a multivariate normal probability for any general culture probabilities. They state that the multivariate normality is achieved by the fact that the multinomial distribution tends to the multivariate normal distribution. The multivariate normality however, is the consequence of the fact that a set of linear combinations of N_i values, defined by equation (2), tends to the multivariate normal distribution by the application of multivariate central limit theorem, as we proceed to show here. In this section we show a rigorous statistical derivation of the limit which covers the cases when n is odd or even.

Most researchers in this area have focused on the odd n (number of voters) case. This is because ties do not occur while determining the majority rule; i.e., for odd number of voters n, the number of voters preferring C_i to C_j will never equal the voters preferring C_j to C_i. However, if n is even there is a positive probability of having a tie. We show, in what follows, that the probability of a tie for n even tends to zero as $n \rightarrow \infty$. Let $p_{ij} = \sum_r p_r$ the with summation extending over the preference rankings in which C_i is preferred over C_j and let T denote the total number of voters preferring C_i to C_j. Then, it is well known from Multinomial distribution theory that T is a Binomial variable with n and p_{ij} as parameters. The probability of a tie in the majority determination between C_i and C_j is

$$\begin{aligned} P(T = \frac{n}{2}) &= \binom{n}{\frac{n}{2}} {p_{ij}}^{\frac{n}{2}} (1 - p_{ij})^{\frac{n}{2}} \\ &= \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} [p_{ij}(1 - p_{ij})]^{\frac{n}{2}} \qquad (4) \end{aligned}$$

We know by Stirling's approximation [5] Page 130, that

$$n! \cong \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}}$$

The symbol $\cong$ means $\frac{n!}{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}} \to 1$ as $n \to \infty$. Applying this to equation (4) and noting that $p_{ij}(1-p_{ij}) \leq \frac{1}{4}$ for all $0 \leq p_{ij} \leq 1$, we can verify that

$$
\begin{aligned}
P(T = \frac{n}{2}) &\cong \frac{1}{\sqrt{2\pi}}\frac{2^{n+1}}{\sqrt{n}}[p_{ij}(1-p_{ij})]^{\frac{n}{2}} \\
&\leq \frac{1}{\sqrt{2n\pi}}\frac{2^{n+1}}{4^{\frac{n}{2}}} \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty
\end{aligned}
$$

For a given voter v, for $i = 1, \cdots, m$, j= $1, \cdots, m$, $i \neq j$, define

$$
\begin{aligned}
X(i,j,v) &= 1 \text{ if } C_i P C_j \\
&= -1 \text{ if } C_j P C_i
\end{aligned} \quad (5)
$$

For the voter profile $(N_1, N_2, \cdots, N_K)$, $C_i M C_j$ if $\sum_{v=1}^{n} X(i,j,v) \geq 1$. The above representation has been used in Bell ([2]) and Gehrlein ([12]). Note that

$$
\sum_{v=1}^{n} X(i,j,v) = a_{1,(i,j)}N_1 + a_{2,(i,j)}N_2 + \cdots + a_{K,(i,j)}N_K
$$

We have,

$$
\begin{aligned}
\lambda_{ij} &= E(X(i,j,v)) \\
&= a_{1,(i,j)}p_1 + a_{2,(i,j)}p_2 + \cdots + a_{K,(i,j)}p_K \quad (6) \\
Var(X(i,j,v)) &= 1 - \lambda_{ij}^2 \quad (7)
\end{aligned}
$$

We also need to obtain the correlation coefficient between $X(i,j,v)$ and $X(i,l,v)$, $i \neq j$, $i \neq l$ and $j \neq l$. To express correlation, we first define

$$\begin{aligned} a_{r,(i,j),l} &= 1 \text{ if in the preference ranking corresponding to } N_r, r = 1, \cdots, K \text{ is} \\ &\quad C_i P C_j \text{ and } C_i P C_l \\ &\quad \text{or } C_j P C_i \text{ and } C_l P C_i \\ &= -1 \text{ Otherwise} \end{aligned} \tag{8}$$

Covariance is defined by the following equation.

$$Covariance(X(i,j,v), X(i,l,v)) = E[X(i,j,v)X(i,l,v)] - \lambda_{ij}\lambda_{il}$$

The correlation between $X(i,j,v)$ and $X(i,l,v)$ is given by

$$\begin{aligned} R_{jl}^{(i)} &= \frac{E(X(i,j,v).X(i,l,v)) - \lambda_{ij}\lambda_{il}}{\sqrt{(1-\lambda_{ij}^2)(1-\lambda_{il}^2)}} \quad j = 1, \cdots, m, \\ &\quad l = 1, \cdots, m \text{ and } j \neq l,\ j \neq i,\ l \neq i. \\ &= \frac{\sum_{r=1}^{K} a_{r,(i,j),l}\, p_r - \lambda_{ij}\lambda_{il}}{\sqrt{(1-\lambda_{ij}^2)(1-\lambda_{il}^2)}} \end{aligned}$$

For a given $i = 1, 2, \cdots, m$, let R_i denote the $(m-1) \times (m-1)$ correlation matrix $(R_{jl}^{(i)})$. Note that $R_{jj}^{(i)} = 1, j = 1, 2, \cdots, m; j \neq i$.

Let

$$\begin{aligned} y(i,j,v) &= \frac{(X(i,j,v)-\lambda_{ij})}{\sqrt{1-\lambda_{ij}^2}}, j=1,2,\cdots,m, j\neq i \\ \text{and} \quad Z_n(i,j) &= \sum_{v=1}^{n} \frac{y(i,j,v)}{\sqrt{n}} \\ &= \sum_{v=1}^{n} \frac{(X(i,j,v)-\lambda_{ij})}{\sqrt{n}\sqrt{1-\lambda_{ij}^2}}, j=1,\cdots,m, j\neq i \qquad (9) \end{aligned}$$

Note that the $(m-1)$ random variables $y(i,j,v), j = 1,\cdots,m, j \neq i$ have a joint distribution with zero mean vector, unit variances and Correlation matrix R_i.

From (9), the probability that C_iMC_j is given by

$$P\left(\sum_{v=1}^{n} X(i,j,v) \geq 1\right) \Longrightarrow P\left(Z_n(i,j) \geq \frac{(\frac{1}{\sqrt{n}} - \sqrt{n}\lambda_{ij})}{\sqrt{1-\lambda_{ij}^2}}\right). \qquad (10)$$

The probability that C_i is the Condorcet winner is therefore given by the joint probability

$$P\left[Z_n(i,j) \geq \frac{(\frac{1}{\sqrt{n}} - \sqrt{n}\lambda_{ij})}{\sqrt{1-\lambda_{ij}^2}}, j=1,\cdots,m, \;\; j\neq i\right]. \qquad (11)$$

By the multivariate central limit theorem (See Cramèr [6], Page 112) applied to independent summands $Z_n(i,j)$,$j = 1,\cdots,m,\ j \neq i$, the joint distribution of

$$(Z_n(i,1),\cdots,Z_n(i,i-1),Z_n(i,i+1),\cdots,Z_n(i,m))$$

tends as $n \to \infty$ to the $(m-1)$ dimensional multivariate normal distribution with zero mean vector, unit variances and correlation matrix $R_i = (R_{jl}^{(i)})$. Let us write

$$\delta_{ij} = \begin{cases} -\infty & \text{if } \lambda_{ij} > 0 \\ \infty & \text{if } \lambda_{ij} < 0 \\ 0 & \text{if } \lambda_{ij} = 0 \end{cases}$$

Denote by $L(h_1, h_2, \cdots, h_{m-1}; R)$ the multivariate normal probability

$$P(Z_1^* \geq h_1, Z_2^* \geq h_2, \cdots, Z_{m-1}^* \geq h_{m-1}; R)$$

where $(Z_1^*, Z_2^*, \cdots, Z_{m-1}^*)$ has the multivariate normal distribution with zero mean vector, unit variances and Correlation matrix R. Then we have

$$\lim_{n\to\infty} P\left[Z_{ij} > \frac{(\frac{1}{\sqrt{n}} - \sqrt{n}\lambda_{ij})}{\sqrt{1-\lambda_{ij}^2}}, j = 1, \cdots, m, \;\; i \neq j\right]$$

$$= L(\delta_{i1}, \delta_{i2}, \cdots, \delta_{i,i-1}, \delta_{i,i+1}, \cdots, \delta_{im}; R_i). \qquad (12)$$

The limiting probability of a Condorcet winner is then

$$P(\infty, m) \;=\; \sum_{i=1}^{m} L(\delta_{i1}, \delta_{i2}, \cdots, \delta_{i,i-1}, \delta_{i,i+1}, \cdots, \delta_{im}; R_i). \qquad (13)$$

Equation (13) shows that the limiting Condorcet winner probability is obtained as the sum of m L-functions. The arguments of the L functions are either $+\infty$ or $-\infty$ or 0. If at least one of the arguments in a L function is $+\infty$, the function value is 0 by the definition of the L-function. We can also drop all the arguments with value $-\infty$ and calculate the reduced L-function from the marginal multivariate normal distribution. It is therefore interesting to note that even for arbitrary probability distribution p_i's, at worst we have to evaluate positive orthant probabilities of a multivariate normal distribution. We shall give examples for the case $m = 3$ and $m = 4$ in later sections and provide a scheme to obtain the exact limiting probability expressions for a Condorcet winner for any choice of p_i's.

Equation (13) can also be specialized to the Dual Culture(DC) Model. A preference rank order is the dual of another ranking if each can be obtained by reversing the order of the other. For example, if $m = 4$, rank order $C_1C_4C_3C_2$ is dual to $C_2C_3C_4C_1$. The DC model assigns p_i such that $p_i = p_j$ if i is dual to j. In this case it is clear from the definition of λ_{ij} that all the $\lambda_{ij} \equiv 0$. This, in turn, implies that $\delta_{ij} \equiv 0$. Equation (13) shows then that the limiting probability of a Condorcet winner under DC models

$$\sum_{i=1}^{m} L(0, 0, 0, \cdots, 0; R_i)$$

Note that in each L function, there are $(m-1)$ zeroes. Gehrlein [11] gives the exact expressions for $m = 3$ and $m = 4$.

3 Condorcet Winner: Some Special Cases

First we assume that all preference rankings for a voter are equally likely. This implies that $p_i \equiv 1/K$, for $i = 1, \cdots, K$. For this case, symmetry conditions imply that $\lambda_{ij} \equiv 0$ for $i = 1, \cdots, m$, $j = 1, \cdots, m$ and $i \neq j$ $\Rightarrow \delta_{ij} \equiv 0$. For this case, it can be verified from [13] that $R_i \equiv R^*$ where R^* is the $(m-1) \times (m-1)$ equicorrelated matrix with equal correlation value $= 1/3$. The limiting probability of a Condorcet winner from (13) is given by

$$P(\infty, m) = mL(0, 0, \cdots, 0; R^*) \tag{14}$$

This expression for $m = 3$ was obtained by Gilbaud ([14]) as $\frac{3}{4} + \frac{3}{2\pi} \arcsin(\frac{1}{3})$. Garman and Kamien [9] gave the expression for $m = 4$ without giving details. For $m \geq 5$, several researchers have obtained approximate values for expression (14) based on simulation studies or by quadrature methods. See for example Gehrlein [13], Jones et al [17] and Niemi et al [23]. The expression (14) shows that the limiting probability for m candidates can be obtained by an evaluation of positive orthant probabilities for a $(m-1)$ dimensional multivariate normal distribution with zero mean vector, unit variances and equicorrelated correlation matrix with all correlations equal to 1/3. Several papers in the statistical literature deal with such evaluations. See, for example, David and Mallows [7], Sondhi[26], Placket[24], Ruben[25], Johnson and Kotz [16] and Bacon [1]. Bacon [1] derives explicit expressions for the positive orthant probabilities for a few small values of m for a multivariate normal distribution with equicorrelated correlation matrix with all correlations equal to ρ. He also derives a recursive relation to calculate the probabilities for successive values of m. Except for a few small values of m, such expressions involve the evaluation of multiple integrals without known closed form expressions. Denote the expression (14) by $P(m)$. Using the expressions from Bacon [1], Gehrlein [10] obtains the values of $P(m)$ for some small values of m as

$m = 3, \quad P(3) = \frac{3}{4} + \frac{3}{2\pi}\arcsin(\frac{1}{3})$

$m = 4, \quad P(4) = \frac{1}{2}[1 + \frac{6}{\pi}\arcsin(\frac{1}{3})]$

$m = 5, \quad P(5) = \frac{5}{16}[1 + \frac{12}{\pi}\arcsin(\frac{1}{3}) + \frac{24}{\pi^2}\int_0^{\frac{1}{3}}\arcsin(\frac{\lambda}{1+2\lambda})\frac{d\lambda}{\sqrt{(1-\lambda^2)}}]$

$m = 6, \quad P(6) = \frac{3}{16}[1 + \frac{20}{\pi}\arcsin(\frac{1}{3}) + \frac{120}{\pi^2}\int_0^{\frac{1}{3}}\arcsin(\frac{\lambda}{1+2\lambda})\frac{d\lambda}{\sqrt{(1-\lambda^2)}}]$

$m = 7, \quad P(7) = \frac{7}{64}[1 + \frac{30}{\pi}\arcsin(\frac{1}{3}) + \frac{360}{\pi^2}\int_0^{\frac{1}{3}}\arcsin(\frac{\lambda}{1+2\lambda})\frac{d\lambda}{\sqrt{(1-\lambda^2)}}$
$\quad + \frac{720}{\pi^3}\int_0^{\frac{1}{3}}\int_0^{\frac{\mu}{1+2\mu}}\arcsin(\frac{\lambda}{1+2\lambda})\frac{d\lambda}{\sqrt{(1-\lambda^2)}}\frac{d\mu}{\sqrt{(1-\mu^2)}}]$

Note however, that some of these expressions involve integrals.

Denote $L(0, 0, \cdots, 0, \rho)$ by $L_{m-1}(\rho)$ when there are $m - 1$ zeros in the argument of L and ρ is the common correlation. From the results of Sampford (See Moran [21]) we have

$$L_{m-1}(\rho) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}(1 - \Phi(at))^{m-1}dt$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution and $a = \frac{2\rho}{1-\rho}$. For the IC model, $\rho = \frac{1}{3}$ and hence $a = 1$. Thus

$$L_{m-1}(0,\cdots,0;\frac{1}{3})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}(1-\Phi(t))^{m-1}dt \tag{15}$$

It is also known that $L_{m-1}(\rho)$ for m being even can be obtained recursively from $L_i(\rho)$ for $i\leq m-2$:

$$L_{m-1}(\rho)=\frac{1}{2}\left[\frac{-(m-1)}{2}+1+\binom{m-1}{2}L_2(\rho)-\binom{m-1}{3}L_3(\rho)+\cdots+\binom{m-1}{m-2}L_{m-2}(\rho)\right] \tag{16}$$

See Johnson and Kotz [16] Pages 48-52 for more details. We obtained the expression for $m=6$ using the recursion formula.

For $m>7$, Bacon [1] provides an approximation believed to be reasonably close to the true value. Figure 1 shows the plot of $P(m)$ for increasing values of m based on these approximations. May [19] has shown that $P(m)\to 0$, as $m\to\infty$ at the rate of $\frac{1}{m}$. His proof is based on saddle point approximation of an integral and it appears to be complicated and difficult to understand. We present a direct proof of this result.

From (14) and (15), we have

$$\begin{aligned} P(m) &= \frac{m}{\sqrt{\pi}}\left[\int_{-\infty}^{\infty}e^{-t^2}(1-\Phi(t))^{m-1}dt\right] \\ &= \sqrt{2}m\left[\int_{-\infty}^{\infty}\phi(t)(1-\Phi(t))^{m-1}e^{\frac{-t^2}{2}}dt\right] \end{aligned}$$

where

$$\phi(t)=(\frac{1}{\sqrt{2\pi}})e^{\frac{-t^2}{2}}=\frac{d}{dt}\Phi(t)$$

$$\begin{aligned} P(m) &= \sqrt{2}m\left[\left\{\frac{-(1-\Phi(t))^m e^{\frac{-t^2}{2}}}{m}\right\}_{-\infty}^{\infty} - \frac{1}{m}\int_{-\infty}^{\infty}[1-\Phi(t)]^m t e^{\frac{-t^2}{2}}dt\right] \\ &= -\sqrt{2}\int_{-\infty}^{\infty}[1-\Phi(t)]^m e^{\frac{-t^2}{4}} t e^{\frac{-t^2}{4}}dt \end{aligned}$$

Applying Cauchy-Schwarz's inequality

$$\left|\int f(t)g(t)dt\right| \leq \left(\int f^2(t)dt\right)^{\frac{1}{2}}\left(\int g^2(t)dt\right)^{\frac{1}{2}} \tag{17}$$

to the above equation, we get

$$P(m) \leq \sqrt{2}\left(\int_{-\infty}^{\infty}(1-\Phi(t))^{2m}e^{\frac{-t^2}{2}}dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}t^2 e^{\frac{-t^2}{2}}dt\right)^{\frac{1}{2}}$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty}(1-\Phi(t))^{2m}e^{\frac{-t^2}{2}}dt &= \sqrt{2\pi}\int_{-\infty}^{\infty}(1-\Phi(t))^{2m}\phi(t)dt \\ &= \frac{\sqrt{2\pi}}{2m+1} \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty}t^2 e^{\frac{-t^2}{2}}dt &= \sqrt{2\pi}\int_{-\infty}^{\infty}t^2\phi(t)dt \\ &= \sqrt{2\pi} \end{aligned}$$

Thus

$$P(m) \leq 2\pi\sqrt{2}\frac{1}{(2m+1)^{\frac{1}{2}}} \rightarrow \ 0 \text{ as } m \rightarrow \infty$$

We proved that the limiting probability $P(m)$ of a Condorcet winner tends to zero as $m \to \infty$. We show in addition that$P(m)$ is monotonically decreasing in m. This settles Kelly's [18] and Buckley and Westen's [4] conjecture for large values of n. This monotonicity property, we believe, is new.

Proposition: $P(m+1) < P(m)$

Proof: From Equations (14) and (15), we have to show that

$$\frac{(m+1)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (1-\Phi(t))^m dt < \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (1-\Phi(t))^{m-1} dt \tag{18}$$

$$\frac{(m+1)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (1-\Phi(t))^m dt = \sqrt{2}(m+1) \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} (1-\Phi(t))^m \phi(t) dt \tag{19}$$

where $\phi(t) =$ pdf of standard normal distribution. The right side of equation (19) equals after integrating by parts

$$-\sqrt{2} \int_{-\infty}^{\infty} t e^{\frac{-t^2}{2}} (1-\Phi(t))^{m+1} dt$$

Similarly the right side of equation (18) equals

$$-\sqrt{2} \int_{-\infty}^{\infty} t e^{\frac{-t^2}{2}} (1-\Phi(t))^m dt$$

Thus it is enough to show

$$\int_{-\infty}^{\infty} te^{\frac{-t^2}{2}} \left((1-\Phi(t))^{m+1} - (1-\Phi(t))^{m}\right) dt > 0$$

Split the region of integration into $\int_{-\infty}^{0} + \int_{0}^{\infty}$,put $t = -y$ in the first integral and after simplification and noting $1 - \Phi(-y) = \Phi(y)$, it reduces to show that

$$\int_{0}^{\infty} \left[\left((1-\Phi(y))^{m+1} - (1-\Phi(y))^{m}\right) - \left(\Phi^{m+1}(y) - \Phi^{m}(y)\right)\right] y\phi(y)dy > 0 \tag{20}$$

We have written $\Phi^{m+1}(y)$ for $\Phi(y)^{m+1}$ and $\Phi^{m}(y)$ for $\Phi(y)^{m}$. We will show that

$$(\Phi^{m}(y) - (1-\Phi(y))^{m}) - \left(\Phi^{m+1}(y) - (1-\Phi(y))^{m+1}\right) > 0 \quad \text{for } y > 0 \tag{21}$$

Write $\Phi(y) = z$ and note that the left hand side of equation (21) equals

$$(2z-1)\left[z^{m-1} + z^{m-2}(1-z) + \cdots + (1-z)^{m-1}\right] - (2z-1)\left[z^{m} + z^{m-1}(1-z) + \cdots + (1-z)^{m}\right] \tag{22}$$

Observe that $2z - 1 > 0$ for $y > 0$.

Note that $z^{m-1} - z^{m-1}(1-z) = z^{m}$. Further,

$$z^{m-2}(1-z) - z^{m-2}(1-z)^2 = z^{m-1}(1-z)$$

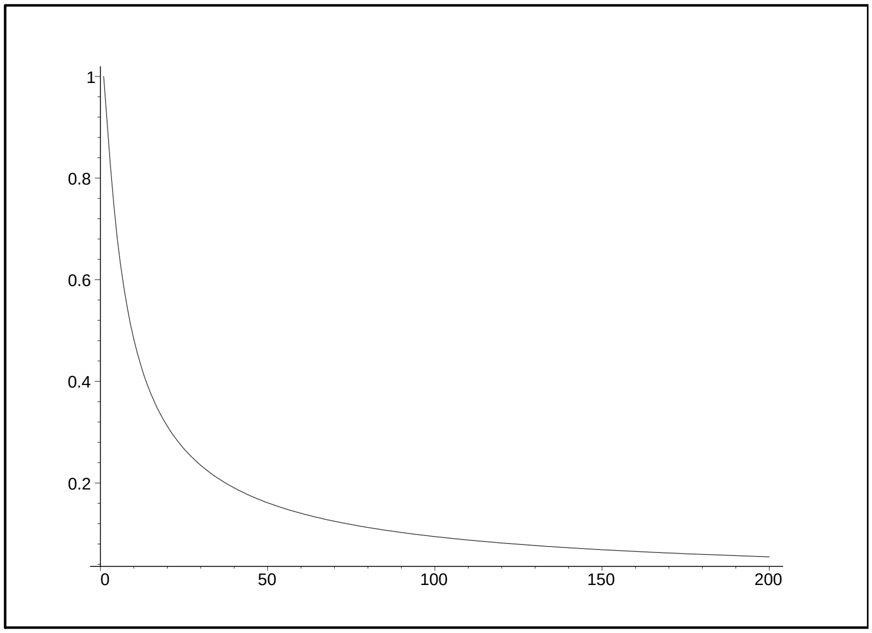

Figure 1: The plot of limiting Probability of a Condorcet winner vs the number of Candidates

and so on.

Hence the expression in the square brackets of equation (20)

$$= z^{m-1}(1-z) + z^{m-2}(1-z)^2 + \cdots + z(1-z)^{m-1} > 0$$

Thus the integrand in equation (20) is >0 and hence the left side of equation (20) is > 0.

4 Condorcet Winner: Special Case $m = 3$ candidates

Niemi et al [23] calculated the limiting probability of a Condorcet winner values for $n = 1..49$, utilizing the table of Ruben [25]. Ruben used quadrature methods and recursive formulas to obtain these values. May [19] and Garman et al [9] discussed the case of 3 candidates to obtain the limiting probability of a Condorcet winner. Gehrlein [11] discusses several possibilities that may arise in obtaining the value for $m = 3$. We give a compact and complete Table for all possible p_i's utilizing the results of Section 2 and (13) of this paper. Suppose the 6 rankings are

Rankings	Number of Voters	Probabilities
$C_1C_2C_3$	N_1	p_1
$C_1C_3C_2$	N_2	p_2
$C_2C_1C_3$	N_3	p_3
$C_2C_3C_1$	N_4	p_4
$C_3C_1C_2$	N_5	p_5
$C_3C_2C_1$	N_6	p_6

It can be verified that

$$\begin{aligned}
\lambda_{12} &= p_1 + p_2 + p_5 - p_3 - p_4 - p_6 \\
\lambda_{13} &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\
\lambda_{21} &= p_3 + p_4 + p_6 - p_1 - p_2 - p_5 \\
\lambda_{23} &= p_1 + p_3 + p_4 - p_2 - p_5 - p_6 \\
\lambda_{31} &= p_4 + p_5 + p_6 - p_1 - p_2 - p_3 \\
\lambda_{32} &= p_2 + p_5 + p_6 - p_1 - p_3 - p_4
\end{aligned}$$

The three correlation matrices R_1, R_2, R_3 are given by:

$$R_1 = \begin{pmatrix} 1 & \frac{(p_1+p_2+p_4+p_6-p_3-p_5)-\lambda_{12}\lambda_{13}}{\sqrt{(1-\lambda_{12}^2)(1-\lambda_{13}^2)}} \\ \frac{(p_1+p_2+p_4+p_6-p_3-p_5)-\lambda_{12}\lambda_{13}}{\sqrt{(1-\lambda_{12}^2)(1-\lambda_{13}^2)}} & 1 \end{pmatrix}$$

The (1,2) entry of R_1 is R_{23}^1.

$$R_2 = \begin{pmatrix} 1 & \frac{(p_2+p_3+p_4+p_5-p_1-p_6)-\lambda_{21}\lambda_{23}}{\sqrt{(1-\lambda_{21}^2)(1-\lambda_{23}^2)}} \\ \frac{(p_2+p_3+p_4+p_5-p_1-p_6)-\lambda_{21}\lambda_{23}}{\sqrt{(1-\lambda_{21}^2)(1-\lambda_{23}^2)}} & 1 \end{pmatrix}$$

The (1,2) entry of R_2 is R_{13}^2.

$$R_3 = \begin{pmatrix} 1 & \frac{(p_1+p_3+p_5+p_6-p_2-p_4)-\lambda_{31}\lambda_{32}}{\sqrt{(1-\lambda_{31}^2)(1-\lambda_{32}^2)}} \\ \frac{(p_1+p_3+p_5+p_6-p_2-p_4)-\lambda_{31}\lambda_{32}}{\sqrt{(1-\lambda_{31}^2)(1-\lambda_{32}^2)}} & 1 \end{pmatrix}$$

The (1,2) entry of R_3 is R^3_{12}. The probability of a Condorcet winner is

$$L(\delta_{12}, \delta_{13}, R^1_{23}) + L(\delta_{21}, \delta_{23}, R^2_{13}) + L(\delta_{31}, \delta_{32}, R^3_{12}). \qquad (23)$$

By considering the 27 possibilities with which $\lambda_{12}, \lambda_{13}$ and λ_{23} can take zero, positive and negative values, we can simplify the expression further. Table 1 shows the 27 possible cases and the corresponding limiting value of the probability of a Condorcet winner. A set of different values is obtained for varying values of p_i's. It is also to be noted that for a few alternatives the values of 0 and 1 are obtained for the limiting probability of a Condorcet winner. See alternatives 7 and 17 in Table 1. Table 1 provides the reference table to calculate the exact limiting probability for any choice of $p_i, i = 1, \cdots, 6$. One of the 27 possibilities will apply in a given situation and one can read off or calculate the value. It is seen from Table 1 that only in two cases out of 27, the limiting probability is definitely 0. In 12 cases (more than 40%) the probability is 1 and in almost all cases the probability will be at least 50%. These observations are consistent with often stated conclusion that for small number of candidates and large number of votes, the Condercet scheme will produce a winner.

It can be verified that case 7, for example, is reached when $p_1 = p_2 = p_5 = \frac{1}{6}$; $p_3 = \frac{2}{5}$ and $p_6 = \frac{1}{10}$. Case 17 is reached when $p_1 = \frac{3}{22}$,$p_2 = \frac{3}{22}$,$p_3 = \frac{9}{44}$,$p_4 = \frac{13}{44}$, $p_5 = \frac{5}{22}$ and $p_6 = 0$. For another set of examples for $m = 3$, see May [19], Garman et al [9] and Meyer et al [20].

For example, the trivial set of probabilities $p_i = 1, p_j = 0, j \neq i$ will produce a Condorcet winner for every n and hence the limit is also 1 as $n \to \infty$. Choose the probabilities p_i such that $C_1C_2C_3C_4 \cdots C_m$, $C_2C_3C_1C_4 \cdots C_m$, $C_3C_1C_2C_4 \cdots C_m$ have probabilities $\frac{1}{3}$ each and the rest of the rankings have probabilities 0. It is easily verified that $\lambda_{12} = \frac{1}{3}$, $\lambda_{13} = \frac{-1}{3}$, $\lambda_{23} = \frac{1}{3}$, $\lambda_{1j} = \lambda_{2j} = \lambda_{3j} = 1$ for all $j \geq 4$. This implies that $\delta_{13} = \infty$, $\delta_{12} = \delta_{23} = -\infty$, $\delta_{1j} = \delta_{2j} = \delta_{3j} = -\infty$, for $j \geq 4$. From equation (12), it follows immediately that in every L term, one of the arguments is a $+\infty$. This implies that the limiting Condorcet winner probability is 0. Thus for every m, we can choose p_i so that the limiting Condorcet winner is 0.

Gillett[15] conjectured that the probability of a Condorcet winner $P(3, n; p)$ for $m = 3$ candidates and any n, the number of voters, is minimized for $p_1 = p_4 = p_5 = \frac{1}{3}, p_2 = p_3 = p_6 = 0$ or $p_1 = p_4 = p_5 = 0, p_2 = p_3 = p_6 = \frac{1}{3}$. Buckley [3] proved this conjecture for the special case $n = 3$. We will show the validity of the conjecture for general n.

The probability of candidate C_1 being the majority winner is

$$P\left\{N_1 + N_2 + N_3 > \frac{n}{2} \text{ and } N_1 + N_2 + N_5 > \frac{n}{2}\right\}$$

$$= P\left\{N_1 + N_2 > \frac{n}{2} - min(N_3, N_5)\right\}$$

$$\geq P\left\{N_1 + N_2 > \frac{n}{2}\right\},$$

since N_3 and N_4 are non-negative random variables. The above expression is greater than or equal to

$P\left\{N_1 > \frac{n}{2}\right\}$, again for the same reason with regard to N_2. N_1 is distributed as a Binomial variable with n and p_1.

$$P\left(N_1 > \frac{n}{2}\right) = 1 - B(k; n, p_1)$$

where $B(k; n, p_1)$ is the Binomial cumulative distribution with parameters n and p_1. $k = \frac{n-1}{2}$ if n is odd and $k = \frac{n}{2}$ if n is even. From the relation of $B(k; n, p_1)$ with Incomplete Beta function, see Feller(1957) page 127, [8],

$$P(N_1 > \frac{n}{2}) = n\binom{n-1}{k} \int_0^{p_1} t^k (1-t)^{n-k-1} dt \tag{24}$$

Continuing similar arguments for C_2 and C_3 being majority winners, we have

$$P(3, n; \mathbf{p}) \geq n\binom{n-1}{k} \left[\int_0^{p_1} t^k (1-t)^{n-k-1} dt + \int_0^{p_4} t^k (1-t)^{n-k-1} dt + \int_0^{p_5} t^k (1-t)^{n-k-1} dt\right]$$

Consider minimizing the following function

$$\begin{aligned} u(p_1, p_4, p_5) &= \int_0^{p_1} t^k (1-t)^{n-k-1} dt + \int_0^{p_4} t^k (1-t)^{n-k-1} dt \\ &+ \int_0^{p_5} t^k (1-t)^{n-k-1} dt \\ \text{subject to } p_1 + p_4 + p_5 &= c \\ p_1 \geq 0, p_4 \geq 0, p_5 \geq 0 \text{ where } c &= 1 - (p_2 + p_3 + p_6) \end{aligned} \tag{25}$$

We proceed to show that an optimal solution to the problem defined by (25) is given by $p_1 = p_4 = p_5 = \frac{c}{3}$.

$u(p_1, p_4, p_5)$ is a separable function in p_1, p_4 and p_5. It can be verified that

$$\frac{\partial^2 u(p_1, p_4, p_5)}{\partial {p_i}^2} = {p_i}^{k-1}(1 - {p_i}^{n-k-2})\left(k - (n-1)p_i\right), i = 1, 4, 5 \qquad (26)$$

For $p_i \leq \frac{k}{n-1}, i = 1, 4, 5, \frac{\partial^2 u}{\partial {p_i}^2} \geq 0 \Longrightarrow u(p_1, p_4, p_5)$ is convex in p_i in the region of p_i's.

In order that p_i's satisfy the constraints of equation(25), we cannot have two of the p_i's are $\geq \frac{k}{n-1}$ with at least one of them $> \frac{k}{n-1}$, since their sum will be greater than $\frac{2k}{n-1} \geq 1$ for both n even and n odd. If for any feasible solution, a single $p_i > \frac{k}{n-1}$, which implies that the other two $< \frac{k}{n-1}$, the Hessian matrix will have its determinant a negative value because of equation (26). Note that the Hessian is a diagonal matrix. Thus such a solution cannot be a local minimum. Therefore we can confine our solution to the region in which all the p_i's $\leq \frac{k}{n-1}$. It is easy to verify that $p_1 = p_4 = p_5 = \frac{c}{3}$ satisfies the Kuhn-Tucker condition for the minimum. Note also that $\frac{c}{3} < \frac{k}{(n-1)}$, since $\frac{3k}{(n-1)} > 1$. Since $u(p_1, p_4, p_5)$ is convex in this region, the solution is a global minimum of $u(p_1, p_4, p_5)$.

Fix now $p_1 = p_4 = p_5 = \frac{c}{3}$ and repeat the whole argument with p_2, p_3, p_6 as variables and consider the minimization problem of minimizing the following function.

$$
\begin{aligned}
w(p_2,p_3,p_6) &= \int_0^{p_2} t^k(1-t)^{n-k-1}dt + \int_0^{p_3} t^k(1-t)^{n-k-1}dt \\
&+ \int_0^{p_6} t^k(1-t)^{n-k-1}dt \\
\text{subject to } p_2+p_3+p_6 &= 1-c \\
p_2 \geq 0, p_3 \geq 0, p_6 \geq 0 \text{ where } c &= (p_1+p_4+p_5) \qquad (27)
\end{aligned}
$$

As before, the minimum solution of equation (27) is $p_2 = p_3 = p_6 = \frac{1-c}{3}$ for $p_1 = p_4 = p_5 = \frac{c}{3}$. From equation (27), we see that the least value of $w(p_3, p_3, p_6)$ is when $c = 1$. Thus the minimum solution is $p_1 = p_4 = p_5 = \frac{1}{3}$; $p_2 = p_3 = p_6 = 0$. Note that our arguments show that $p_2 = p_3 = p_6 = \frac{1}{3}$; $p_1 = p_4 = p_5 = 0$ is also optimal. Denote by $\tilde{\mathbf{p}}^*$ the solution ${p_1}^* = {p_4}^* = {p_5}^* = \frac{1}{3}$; ${p_2}^* = {p_3}^* = {p_6}^* = 0$. Let $u^*({p_1}^*, {p_4}^*, {p_5}^*)$ be the value of the objective function of (25). We have shown that for any choice of $\tilde{p}$

$$
\begin{aligned}
P(3,n;\tilde{p}) &\geq n\binom{n-1}{k}u^*({p_1}^*, {p_4}^*, {p_5}^*) \\
&= 3(1-B(k;n,\frac{1}{3})), \text{ from (22)}
\end{aligned}
$$

We will now show that $P(3,n;\tilde{\mathbf{p}}^*) = 3(1-B(k;n,\frac{1}{3}))$ which concludes that $\tilde{\mathbf{p}}^*$ yields the minimum value of $P(3,n;\tilde{p})$. Since ${p_2}^* = {p_3}^* = {p_6}^* = 0$, we have

$$
P(3,n;\tilde{\mathbf{p}}^*) = P(N_1 > \frac{n}{2}, N_1 + N_5 > \frac{n}{2})
$$

$$
\begin{aligned}
&+ \quad P(N_4 > \frac{n}{2}, N_4 + N_1 > \frac{n}{2}) \\
&+ \quad P(N_5 > \frac{n}{2}, N_5 + N_4 > \frac{n}{2}) \\
&= \quad P(N_1 > \frac{n}{2}) + P(N_4 > \frac{n}{2}) + P(N_5 > \frac{n}{2}) \\
&= \quad 3(1 - B(k; n, \frac{1}{3}),
\end{aligned}
$$

since N_1, N_4, N_5 are identically Binomial distributed with parameters n and $\frac{1}{3}$.

This establishes the conjecture. Note that the minimum value for the probability of a Condorcet winner is given by

$$3\left(1 - B(k; n, \frac{1}{3})\right)$$

or an equivalent integral representation

$$3n\binom{n-1}{k}\left[\int_0^{\frac{1}{3}} t^k(1-t)^{n-k-1}dt\right]$$

The former is easier to evaluate for small values of n and employ a normal approximation to the Binomial for large n. For example, when $n = 3$, the minimum probability is

$$3\left(\binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \binom{3}{3}\left(\frac{1}{3}\right)^3\right) = \frac{7}{9}$$

which is about 77%. For $n = 4$, the minimum probability of a Condorcet winner is

$$3\left(\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)+\binom{4}{4}\left(\frac{1}{3}\right)^4\right)=\frac{1}{3}$$

which drops down to 33%.

The foregoing arguments can be generalized to $m > 3$ candidates. Consider the probability distribution $\tilde{\mathbf{p}}^*$ on $m!$ preference rank orders given by assigning probability of $\frac{1}{m}$ to each of the cyclic rank orders

$$\begin{array}{rl}
\text{rank order } 1, 2, \cdots, (m-1), m & \text{probability } p_1 \\
\text{rank order } 2, 3, \cdots, m, 1 & \text{probability } p_2 \\
\text{rank order } 3, 4, \cdots, 1, 2 & \text{probability } p_3 \\
\cdots & \cdots \\
\text{rank order } m, 1, \cdots, (m-2), (m-1) & \text{probability } p_m
\end{array} \tag{28}$$

We will establish that this solution yields the minimum for the probability of a Condorcet winner $P(m, n; \tilde{p})$ for a general probability distribution $\tilde{p}$. We use the same approach that was used for $m = 3$ case. It is shown first along similar lines that

$$P(m,n;\tilde{p}) \geq P(N_1 > \frac{n}{2}) + P(N_2 > \frac{n}{2}) + \cdots + P(N_m > \frac{n}{2}) \tag{29}$$

$$= n\binom{n-1}{k}\left[\sum_{i=1}^{m}\int_0^{p_i} t^k(1-t)^{n-k-1}dt\right] \tag{30}$$

From an argument similar to the case $m = 3$, it is shown that

$$P(m, n; \tilde{p}) \geq m(1 - B(k; n, \frac{1}{m})). \tag{31}$$

Next it follows that from (28) and the definition of a Condorcet winner, $P(m, n; \tilde{\mathbf{p}}^*)$ for $\tilde{\mathbf{p}}^*$ given by $\mathbf{p_i}^* = \frac{1}{m}$, for $i = 1, \cdots, m$ and $\mathbf{p_i}^* = 0$ for $i \geq (m+1)$ the probability of C_1 winning is

$$\begin{aligned} P(N_1 + N_3 + \cdots + N_m > \frac{n}{2}, N_1 + N_4 + \cdots + N_m > \frac{n}{2}, \\ \cdots, N_1 > \frac{n}{2}) \\ = P(N_1 > \frac{n}{2}) \end{aligned}$$

Similarly it is verified that the probability of C_2 winning is

$$\begin{aligned} P(N_1 + N_2 + N_4 + \cdots + N_m > \frac{n}{2}, N_1 + N_2 + N_5 + \cdots + N_m > \frac{n}{2}, \\ \cdots, N_2 > \frac{n}{2}) \\ = P(N_2 > \frac{n}{2}) \end{aligned}$$

and so on. Hence

$$\begin{aligned} P(m, n; \tilde{\mathbf{p}}^*) &= P(N_1 > \frac{n}{2}) + P(N_2 > \frac{n}{2}) + \cdots + P(N_m > \frac{n}{2}) \\ &= m(1 - B(k; n, \frac{1}{m})) \end{aligned}$$

which equals the bound in (31). Hence $\tilde{\mathbf{p}}^*$ yields the minimum for the probability of a Condorcet winner. Table 2 gives the minimum probability of a Condorcet winner for selected values of m and n.

5 Condorcet Winner: Special Case m=4 candidates

For this case, we show that it is possible to obtain, for any choice of $p_i, i = 1, 2, \cdots, 24$, exact limiting probability of a Condorcet winner. We first calculate the six quantities $\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}$ and λ_{34} from (6). Then define $\delta_{12}, \delta_{13}, \delta_{14}, \delta_{23}, \delta_{24}$ and δ_{34} as stated in Section 2. Also calculate the correlation matrices $R_i, i = 1, 2, 3, 4$. In this case, there are $3^6 = 729$ possibilities for the λ_{ij}'s to be > 0 or < 0 or $= 0$. It will be unwieldy to prepare a Table containing all these 729 cases. We show how the limiting probability can be calculated for any given situation. The limiting probability of Condorcet winner follows from (13).

$$
\begin{aligned}
P(\infty, 4) &= L(\delta_{12}, \delta_{13}, \delta_{14}; R_1) + L(\delta_{21}, \delta_{23}, \delta_{24}; R_2) \\
&\quad + L(\delta_{31}, \delta_{32}, \delta_{34}; R_3) + L(\delta_{41}, \delta_{42}, \delta_{43}; R_4)
\end{aligned}
$$

Consider the calculation of $L(\delta_{12}, \delta_{13}, \delta_{14}; R_1)$. Calculations for other L-factors are similar. To obtain this, we need at the worst to calculate $L(0, 0, 0; R_1)$ or $L(0, 0, R^1_{ij})$ or $L(0)$. Recall the definition of R^1_{ij} as appropriate i, j element of R_1. Note that $L(0) = \Phi(0) = \frac{1}{2}$, where Φ is the cdf of the standard normal distribution. Bacon [1] gives exact expressions for those L functions as

$$
\begin{aligned}
L(0,0,R^1_{ij}) &= \frac{1}{4} + \frac{1}{2\pi} arcsin(R^1_{ij}) \\
L(0,0,0,R_1) &= \frac{1}{8}\left[1 + \frac{2}{\pi}\left(arcsin(R^1_{23}) + arcsin(R^1_{24}) + arcsin(R^1_{34})\right)\right]
\end{aligned}
$$

The calculation of this L for other possible arguments is by inspection or it will reduce to one of the above 3 cases with arguments equal to zero.

This example illustrates the comment made immediately after equation (13) that the only calculations, if any, involve calculating the positive orthant probabilities of appropriate multivariate normal distribution. Thus for any one of the 729 possibilities that is reached in a given case, we can calculate the exact limiting probability of a Condorcet winner.

6 Conclusion

In this paper, we provide a rigorous mathematical treatment for calculating the limiting probability of a Condorcet winner for any number of candidates and with arbitrary rank order probabilities, when the voters are independent. We show further that the limiting probability depends only on the positive orthant probabilities of an appropriate multinormal distribution even for arbitrary probabilities of the preference rank order probabilities. We provide approximate limiting probabilities for a Condorcet winner for any number of candidates and equal rank order probabilities. We provide a compact and complete Table for the limiting probability of a Condorcet winner with 3 candidates and arbitrary rank order probabilities. We present a simple proof of a result of May [19] to show that the limiting probability of a Condorcet winner tends to zero as the number of candidates tends to infinity. We also present a scheme for calculating the exact limiting probability for $m = 4$ candidates and arbitrary probability distributions on the preference rank orders. We show for the first time that the limiting probability of a Condorcet winner for any given number of candidates m is monotone decreasing in m for the equally likely case. This, in turn, settles the conjectures of Kelly [18] and Buckley and Westen [4] for the case $n \to \infty$. We prove the validity of Gillett's [15] conjecture on the minimum value of the probability of a Condorcet winner for $m = 3$ and any n. Extending the result we obtain the minimum value and the minimum solution for any n and m.

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Number	λ_{12}	λ_{13}	λ_{23}	Probability of a Winner
1	0	0	0	$\frac{1}{4} + \frac{1}{2\pi} \arcsin(R^1_{23}) + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^2_{13}) + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^3_{12})$
2	0	0	+	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^1_{23})$
3	0	0	-	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^1_{23})$
4	0	+	0	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^2_{13})$
5	0	-	0	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^2_{13})$
6	0	+	-	$\frac{1}{2}$
7	0	+	+	1
8	0	-	+	$\frac{1}{2}$
9	0	-	-	1
10	+	0	+	$\frac{1}{2}$
11	+	+	0	1
12	+	0	-	1
13	+	-	0	$\frac{1}{2}$
14	+	+	+	1
15	+	-	-	1
16	+	+	-	1
17	+	-	+	0
18	+	0	0	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^3_{12})$
19	-	0	0	$\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi} \arcsin(R^3_{12})$
20	-	0	+	1
21	-	0	-	$\frac{1}{2}$
22	-	+	0	$\frac{1}{2}$
23	-	+	+	1
24	-	+	-	0
25	-	-	0	1
26	-	-	+	1
27	-	-	-	1

Table 1: Limiting Condorcet Winner Probabilities for $m = 3$ candidates

n	$m = 3$	$m = 4$	$m = 5$	$m = 10$
3	0.7778	0.6250	0.5200	0.2800
4	0.3333	0.2031	0.1360	0.0370
5	0.6296	0.4141	0.2896	0.0856
6	0.3004	0.1504	0.0848	0.0127
7	0.5199	0.2822	0.1667	0.0273
8	0.2638	0.1092	0.0520	0.0043
9	0.4345	0.1957	0.0979	0.0089
10	0.2297	0.0789	0.0318	0.0015
19	0.1943	0.0356	0.0079	0.0000
20	0.1129	0.0158	0.0028	0.0000
29	0.0934	0.0071	0.0007	0.0000
30	0.0564	0.0033	0.0003	0.0000
50	0.0148	0.0002	0.0000	0.0000
51	0.0207	0.0002	0.0000	0.0000
100	0.0006	0.0000	0.0000	0.0000
101	0.0008	0.0000	0.0000	0.0000

Table 2: Minimum Condorcet Winner Probabilities for some values of m and n