A COMPARISON OF THE QUALITY OF LIGHT DISTRIBUTION IN DIFFERENT DIAMOND MODELS

By

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Andrew Faulds optimized the photorealistic rendering code and multi-threaded both the photon shooting and gathering code for a class project.

ABSTRACT

In this thesis, we explore the way different cuts of diamonds affect the light that interacts with them. While some have analyzed diamond shape mathematically, no currently published research has used simulation to determine how light interacts with diamond models. We used forward ray tracing to simulate photons traveling through the scene using reflection and refraction.

We created visualizations to show both single photons traveling through the scene and many photons in a single scene. Different colors were used to differentiate photon wavelengths or to show the number of bounces a photon had taken along its path. Small normal lines were used to show when the photons had hit a diamond face or a wall, which was useful both for debugging purposes on individual hits and when looking at the distribution of many photon hits in a scene. In order to be able to view more photon paths in a scene without them blocking each other, We also created visualizations with low alpha photon paths.

We used forward raytracing to recursively send photons through a scene containing a diamond. The Sellmeier equation was used to determine if a photon was absorbed, and then the Fresnel equation determined if the photon was reflected or refracted. If the photon reflected, we used the reflection equation, and if it refracted, we used Snell's Law to determine the refraction angle.

We found that the two round brilliant diamond models we used performed significantly better than the other types of models at sending photons out of the top half of the model. The princess cut model also did much better than the sphere, which was used as an example of how much light would be sent out of the top if no effort went into designing a diamond to send light out of the top. We used the angle at which the photons leave the diamond and the angle of the faces that they leave from to determine which diamond models perform the best.

1. INTRODUCTION

The way light is affected by the shape of gemstones is an integral part of their look and feel. Therefore, information about how light travels inside them is crucial to have when they are designed and cut. The light coming out of the gem is what causes it to sparkle, and thus an important part of how it will look. When choosing a cut for a stone for a piece of jewelry, it would be helpful for a jeweler to be able to simulate what their piece would look like before actually getting a gem of a certain cut.

The art of shaping diamonds can be traced back many centuries in India and Germany. In the earliest described shaped diamonds, they were cut with faces and were often asymmetrical in shape; the main focus was on removing as little of the diamond as possible. This changed in the 15th century when people started wearing diamonds. A pendeloque shape (a pear-shaped round cut) was used initially, but was abandoned for a cut called a rose cut in the sixteenth century because the latter had a smaller loss in diamond size and a more pleasing light distribution. The first brilliant cut diamonds were made in the seventeenth century, but they had many fewer faces than the brilliant cuts today [1]. Over the years the number of facets, thickness of the diamond, and angles between the faces have been modified to create the round brilliant diamonds used today.

This thesis focuses on simulating how light will react with particular cuts of diamonds. The tool that we wrote for this thesis looks at different existing cuts to compare them and their effect on light. We measured the visual appeal of light as the percentage of light that leaves the top and upper portions of the gem. The research question we are trying to answer is "Using this criteria, which cut of gem is the most visually appealing?"

In ray tracing, a ray is sent out from the camera into the scene, and it bounces based on the ray's reflection and refraction direction. We used a specific type of ray tracing called photon mapping, which is also called forward ray tracing. This is because the rays (i.e. photons) start from the light source as they would in the real world and move through the scene as actual photons would, as opposed to backward ray tracing where rays start from the camera.

Unlike backward ray tracing, forward ray tracing can render caustics and refractive properties [2]. A caustic is a concentration of light caused by light reflecting or refracting specularly on curved surfaces. An example of a refractive property is a rainbow caused by the separation of wavelengths by refraction. Backward ray tracing is still useful because only light that goes into the camera is modeled, and it is also more efficient.

One of the properties of light that changes the ray tracing behavior is the wavelength of the light. Wavelength in light is the distance from peak to peak in a light wave. The wavelengths of light that we can see and that we used in our simulation range from 380 nm to 780 nm. Refractive index is a property of a material that determine how much light refracts when entering that medium. When calculated, the relative refractive index to the material that the light is leaving must also be taken into account. This property is also a function of wavelength. We implemented refractive index to be relative to wavelength because as light refracts in real life we see evidence of this. An example of this is visible rainbows. We also used unpolarized light in all of our testing, which means that the light waves are oriented in all directions. This is because we were simulating indoor lighting which is not polarized.

We used forward ray tracing to simulate light in the scene. Each photon was assigned a wavelength which was used in the calculation of the angle of refraction for that photon. Since the refractive index changes with wavelength, this affects the photon's behavior. We visualized the photons' landing locations and paths. We ran simulations that used a large number of photons and analyzed them to determine the best diamond cut of the cuts we have measured. We chose the best diamond cut as the diamond that sent most light out of the top half of the diamond, because this is what the cuts intend to do. These are the most sparkly diamonds [1].

First we will go over related work and ray tracing in Chapter 2. In Chapter 3, we describe the implementation of our system and how we determined the reflection and refraction of the photons. We created several visualizations for different purposes, which we describe in Chapter 4. In Chapter 5 we describe the models we used and show the data that we collected for the angle at which the photons left the diamond and the angles of the faces out which they left. We describe how this data shows Ideal Diamond model to be the best of the ones that we tested.

2. RELATED WORK

One method of rendering light distributions is radiosity. For radiosity, each face accumulates light that hits it, and then sends the amount of light that hit it into the scene. To determine where the light goes, form factors are determined for each face to determine what percentage of rays coming from one face would hit the other. No information about the incoming direction is stored and light leaves each face at a random direction, so this only works for situations of entirely diffuse light [3]. Since in our system we used diamonds, which have specular reflections, we could not have used radiosity.

In his paper [4], Whitted used the concept of sending rays from the viewer into a scene for a lighting model. While this does allow for both specular and diffuse lighting, it still does not allow refraction to happen, which was crucial for our system with diamonds. In our system, we use a technique like the first part of Jensen's paper [2], where he creates a photon map and then also creates a photorealistic rendering basic on the photon hits. To create the photon map, Jensen uses forward ray tracing sending photons recursively through a scene using the reflection equation, as we did. This is described in more detail in Chapter 3.

Guy and Solar describe in [5] their method of rendering gemstones. It uses graphics hardware to speed up the process and make it render at an interactive rate, but it must sacrifice some of the realism by using simplifications of real physics. Some of these simplifications include only using three wavelengths of light, used a looked table for Fresnel values, and//TODO: more simplifications. Though they use these simplifications, they create very realistic looking renderings of colored and clear gemstones in real time. This paper brings up the point that polarization could play a role in the appearance of light in gemstones. It also shows evidence that after the second bounce of internal reflection, there is little visible change in the light pattern. We did not need to be so restrictive of the number of bounces because it cost us no implementation effort to allow more internal bounces; it mattered more for them because they were working in real time. Even though their results look realistic, we were trying to find which diamond cuts worked better in in as realistic a simulation as possible, so their approach would not work for the problem we are trying to solve.

Diamond Design: A Study of the Reflection and Refraction of light in a Diamond [1] is an in-depth study of many aspects of diamonds. It goes through how and why diamond designs evolved throughout history. It also uses a mathematical approach to take an in-depth look at the first and second bounces in the lower portion of the diamond to demonstrate how the light comes back out again through the top. It proposed what it considers to be some ideal proportions of a diamond.

The Gemological Institute of America (GIA) examines in [6] how the design and cut of modern diamonds has changed over time. More than just proportion changes in the diamond, it also describes how facets on the sides of diamonds have changed size. This is not something we have covered in the scope of our paper.

3. ALGORITHM AND IMPLEMENTATION

We implemented photon mapping and recursively traced each photon independently. It was written in C++ using OpenGL by building off of an existing rendering engine used in the Advanced Computer Graphics course at Rensselaer Polytechnic Institute. We will describe in this chapter how each photon bounces and how those bounces are calculated. We define a photon bounce as when a photon meets a surface and interacts with it. We will discuss how the material properties are used to decide whether the photon will reflect or refract, and are also used to determine the angle at which the photon will bounce in.

Photons travel from the light source either in a specified or a randomly generated direction. When a photon hits a surface, it either reflects, refracts, or is absorbed based on the properties of the material, photon, and hit as described later in this chapter. If the photon continues, the angle it leaves the surface is calculated, and the photon starts the bouncing process again. It continues until it has either left the scene, been absorbed, or reached a set maximum depth. Once a photon has left the scene or is absorbed, it no longer has an effect on the scene, so we can stop keeping track of it. The maximum depth keeps photons from monopolizing the processing time for efficiency. Guy and Solar [5] showed that there is not a big difference in appearance inside the gemstone after two internal bounces. We used a maximum depth of six in our tests because real-time interactivity was not one of our goals.

3.1 Photon Bounces

To send a photon on a bounce, we pass its position, direction, and various other parameters into a recursive method. We perform an intersection test with each quad in the scene, using the photons' position as a starting point for a ray in the direction the photon is going. A ray's starting point is either where the photon originated or the point of its last bounce. The photon travels away from the starting point. An intersection is when two objects in 3D space touch each other. A ray/face intersection occurs when at some point along the ray, the ray passes through or touches the quad in 3D space. Currently all of our models have less than three hundred faces, so we are not using a spatial data structure to cut down on the triangle intersection time. A spatial data structure stores items in such a way to indicate spatial relationships. They reduce the amount of time it takes to find objects in a certain area and near other objects. After all of the ray/face intersections are determined, we determine the one closest to the origin of the ray. That is the face that the photon will intersect with. This process of finding the closest intersection to the beginning of the ray must happen for each bounce of each photon. This is very similar to traditional ray tracing except the photons are traveling in the direction of light travel in the scene as opposed to in the opposite direction of light travel.

Image 3.1 shows a photon traveling within a diamond through multiple bounces. The photon first refracts into the diamond, then reflects off of the bottom surfaces of the diamond.

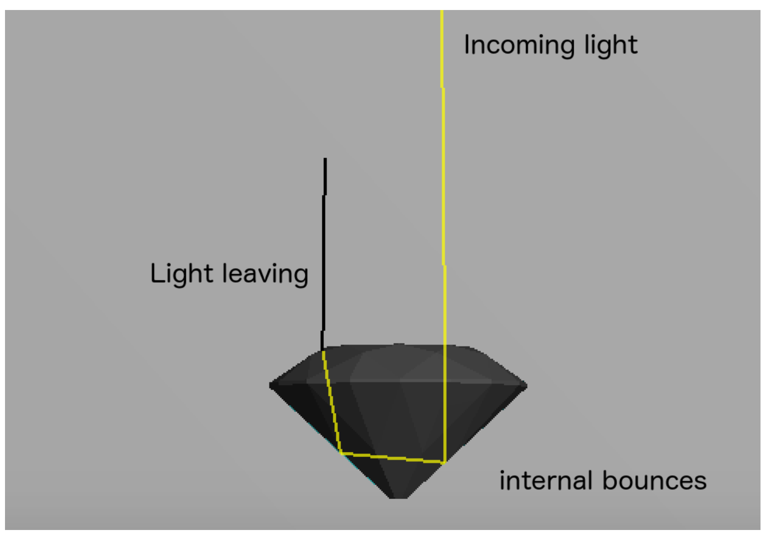

Figure 3.1: This figure shows a single photon's path as it goes through the diamond. It comes in, bounces off of some of the bottom faces, and then comes back out again though the top. It also shows how the shape of the diamond helps get photons to come back out of the top of the shape.

3.2 Material-Photon Interaction

In our system, the absorbency of a material is stored as a property of that material as a value between zero and one. To determine if a photon is absorbed, we generate a random number between zero and one. If it is less than the absorbency value, the photon is absorbed, and does not continue on. Otherwise, it continues as usual. This method of calculating photon absorption or bounce direction is called Russian Roulette. We used 0.1 as our absorbency value for diamonds because it was within the range of absorbency values found for diamonds in the CVD Diamond Booklet [7]. In actuality, the absorbency value of a material depends on many factors such as heat, incoming angle of the photon, and wavelength.

The option that is not Russian Roulette would be to have the photon split and continue in both the reflection and refraction directions with a lower 'energy' value. That can lead to a longer running time for the same number of initial photons because the number of photon hits doubles every bounce [2]. The roulette implementation more accurately reflects what happens in the real world because when a photon hits a surface, it either reflects or refracts. With a sufficiently large number of samples, Russian Roulette should produce the same results as a photon-splitting approach. Do to the set 0.1 absorbency value, the calculation that determines whether or not a photon is absorbed is based on a single value for all colors, not each photon's individual color. The more physically correct thing to do would be to have the wavelength of the photon impact the whether or not it is absorbed.

To find if a photon reflects or refracts, we used the Fresnel equation [8]. In the following equations 3.1 and 3.2, n_1 and n_2 are the indices of refraction of the medium being left and entered, respectively. The values of θ_1 and θ_2 are the angles between the normal of the face and the direction the photon is hitting the face and leaving the face, respectively. The R_s and R_p values in equations 3.1 and 3.2 determine the percentage of photons that will reflect. R_s is the equation for s-polarized light; R_p is the equation for p-polarized light. The difference between s-polarized and p-polarized light, is in the direction of the electric field in relation to the direction of light and the normal of the face the light is striking. We used the index of refraction of air as 1.000293 and calculated the index of refraction of diamonds in equation

3.5.

$$R_s = \left| \frac{n_1 \times \theta_1 - n_2 \times \theta_2}{n_1 \times \theta_1 + n_2 \times \theta_2} \right|^2 \tag{3.1}$$

$$R_p = \left| \frac{n_1 \times \theta_2 - n_2 \times \theta_1}{n_1 \times \theta_2 + n_2 \times \theta_1} \right|^2 \tag{3.2}$$

Since we are using unpolarized light, to get the reflectance value we take the average of the reflectance of the s-polarized and p-polarized light values. The reflectance value is how likely a particular photon is to reflect versus refract. It is different for each hit because it depends on the material that the photon is traveling through, the material the photon intersects with, and the incoming angle with the intersecting surface. Therefore, it must be calculated every time a photon hits a surface.

3.3 Direction of Photon Travel

Depending on what a photon is doing, we must treat the direction of photon travel differently. If a photon is absorbed, very little computation is required because the photon does not bounce. The direction of photon travel is calculated differently if a photon is reflecting or refracting. If a photon is reflecting, when it hits a surface it reflects across the normal of the face it hit according to equation 3.3. In this equation, r is the direction of reflection. The variable i is the incoming ray of light, and n is the normal of the material that the ray is reflecting off of. A diagram to make this formula clearer is in Figure 3.2.

$$r = i - 2(i \cdot n)n \tag{3.3}$$

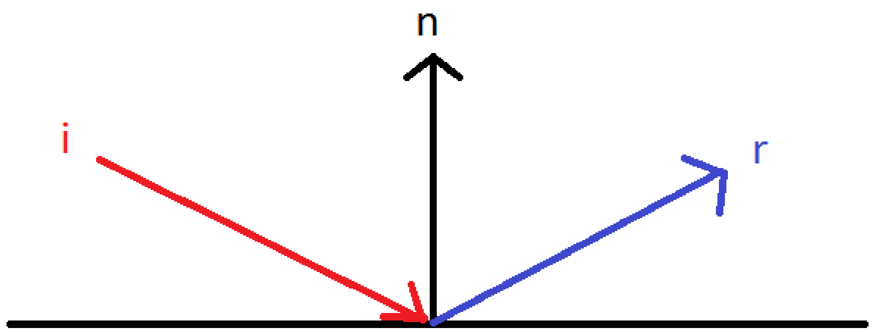

Figure 3.2: This diagram shows an incoming ray, i, reflecting off a surface with the normal n. The ray r is the photon that leaves the surface after the reflection.

The reflection equation holds true for whether a photon is inside an object and is hitting the surface between the object and air from the inside or if the photon is currently in air and is hitting the surface from outside the object. Then we simply trace the photons recursively with the newly calculated photon direction.

The calculation for photon refraction is more complicated. We calculate a photon's refraction direction with Snell's law, equation 3.4 [9], but for this, we need the index of refraction of the medium the photon is currently traveling through and the material that the photon is hitting.

$$n_1 \times \sin\theta_1 = n_2 \times \sin\theta_2 \tag{3.4}$$

Since index of refraction changes with the wavelength of the photon, in equation 3.5, we calculate the index of refraction for each hit individually with the Sellmeier equation. We use the B and C values to calculate the specific index of refraction for that wavelength, where the B and C are constants that are stored as a property of the material in question. We got these values from the CVD Diamond Booklet [7]. The constants are $B_1 = 0.3306$, $B_2 = 4.3356$, $C_1 = 30625$, $C_2 = 11236$, and B_3 and C_3 are not significant. We implemented this feature because we thought that the different wavelengths would make a significant difference in the result. In actuality, we could see no pattern in the different wavelengths of photons when they were sent through the diamonds.

$$n(\lambda) = \sqrt{1 + \frac{B_1 \times \lambda^2}{\lambda^2 - C_1} + \frac{B_2 \times \lambda^2}{\lambda^2 - C_2} + \frac{B_3 \times \lambda^2}{\lambda^2 - C_3}} \tag{3.5}$$

To find the exact angle of refraction, the equation 3.6 from *Reflections and Refractions in Ray Tracing* [10] is used. A provided snippet of code from this paper was used in the calculation of the angle.

$$\sin^2 \theta_2 = \left(\frac{n_1}{n_2}\right)^2 \times (1 - \cos^2 \theta_1) \tag{3.6}$$

3.4 Summary

For our implementation, we used photon mapping. We traced each photon through the scene recursively and separately from the others. Photons travel along a ray until they hit an object, and then they either reflect, refract, or are absorbed. We used the Russian Roulette method for photon bounces, so whenever a photon hits a surface, only one photon continues. We used the Fresnel equation to determine if the photon would reflect or refract, and Snell's law to determine the angle of refraction. We used the Sellmeier equation to determine the index of refraction at each wavelength.

4. VISUALIZATION

While creating our tool, we made visualizations for several different reasons. Some of our visualizations were for ourselves to show that we had done a piece of code correctly. When we had bugs in our code, these visualizations showed us that our code wasn't working as intended, and we used those for debugging and also created new ones for debugging purposes. Some visualizations helped to make better sense of the data we collected.

Being able to send a single photon into the scene was especially helpful with debugging. Using black lines to draw paths of photos escaping the scene was also helpful to know what was happening to photons for debugging. We also used colored paths to help show where photons were after a certain number of bounces. Drawing small lines of the normals of the faces where photons hit helped debug issues with using inverted normals. It also is a helpful visualization to see the spread of the photons hitting the box that can be less cluttered than when the paths are drawn, which is show in Figure 4.1.

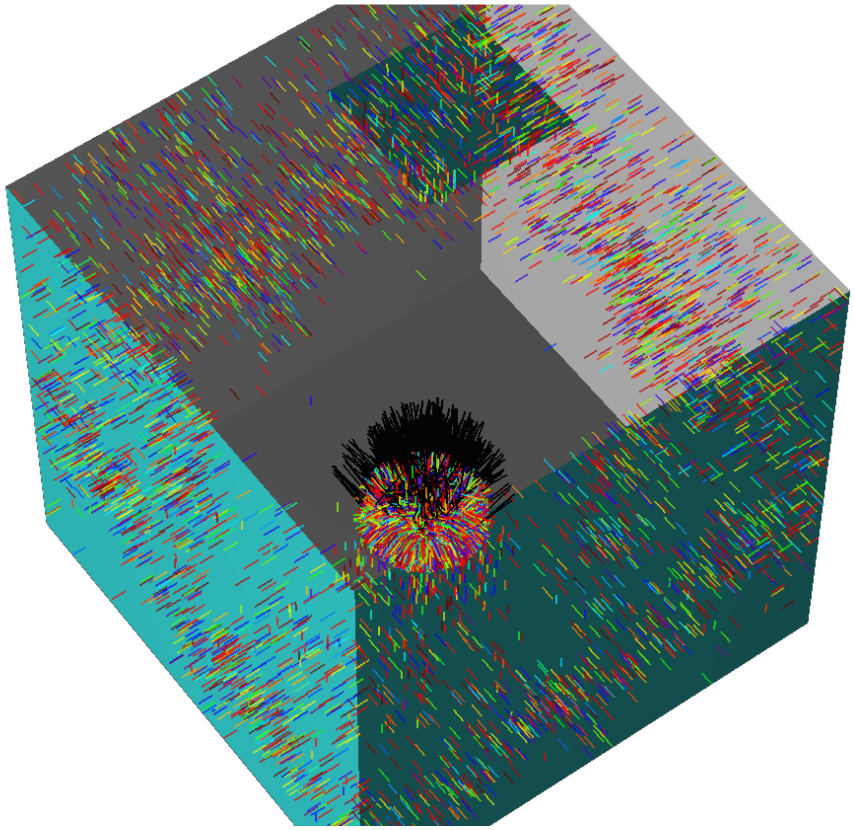

Figure 4.1: This image shows a visualization of where photons landed. Only the normals are drawn so that it is easier to see the distribution of where the photons have landed. We placed the diamond inside of a box with no ceiling so there is a surface to view the photon distribution, and a way to see into the box. There is also a light source in the top right corner of the box. Since the box only absorbs photons and does not reflect any, it has no effect on the photons in the diamond, because photons that hit the box would have left the scene. The colored lines on the walls and floor of the box represent when a photon hit that wall and the colors themselves are an RGB representation of the wavelengths of those photons. The black lines show where a photon has left the scene without touching a wall of the box. Here we see that there is not an even distribution of photons. On the walls of the box there are more colored lines on the upper half than lower half.

4.1 Photon Bounces

The system has many options to visualize the photons bouncing around in the scene. Whether there are many photons or just one, there are several visualizations that are useful in both cases. The normal of the face can be drawn where the photon hits. This shows how the photon refracts towards the normal of the face. It also shows which side of the face the photon is hitting and which side's normal is being used in the refraction and reflection calculations. Usually when light leaves the scene it no longer has an effect on the system, so those photons are not drawn. We have implemented a setting to see those photons for both debugging and visualization purposes, as seen by the black lines in Figures 4.1 and 4.2. Figure 4.1 also uses the normal lines where photons hit to help show the distribution of photons in the scene. There are clearly more normal lines on the higher portions of the walls than on the lower portions and on the bottom face of the box.

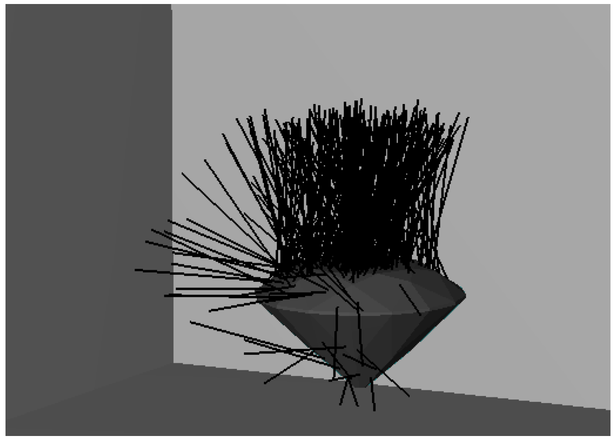

Figure 4.2: In this image photons are coming from the light above the diamond. They are leaving the diamond in all directions, but only some of the photons' bounces are drawn. Here we only show the last bounce of photons as they leave the scene through the missing faces of the box. All other photons and photons bounces are not shown, including the photons that would hit the walls. The exiting photons are leaving the diamond from the top and the front where the box walls have been removed and are visualized with black lines indicating their direction of travel. From this image, we can see how many of the photons are coming out of the top of the diamond. The density of the black lines at the top of the diamond is much greater than the density of the black lines coming out of the side of the diamond not facing a wall. This was a general visualization that showed us that our instincts were correct that more photons were leaving out of the top of the diamond. More detailed and numeric proof can be found in Chapter 5.

4.2 Single Photon

The option to visualize a single photon bouncing is shown in Figure 4.3. To create this visualization, a ray is traced from the current camera location through a set point into the scene. The camera location is the point in space that the scene is viewed from. It can be modified to view the scene from different angles. This is good for seeing how individual rays move around the scene, but does not give an overall feel for where light will gather and the patterns of light in the scene. It is useful for understanding the paths that photons take through the diamond when entering from different locations. It was also a useful tool for debugging the raytracer.

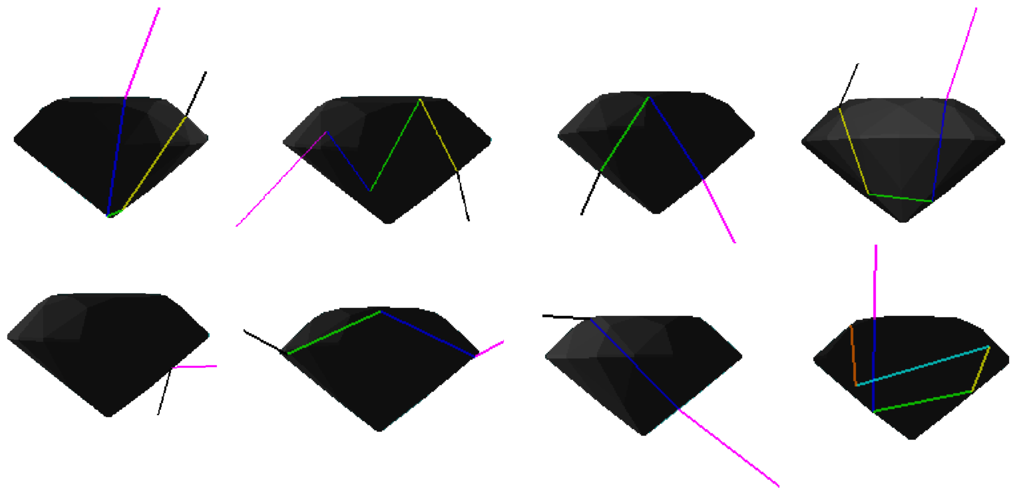

Figure 4.3: This image shows several different paths that photons took through the diamond in our simulations. The magenta line shows how the photon initially hit the diamond, and all other lines show subsequent reflective and refractive bounces. The black lines show how the photons left the diamond. In the bottom right path in this image we see that there is no black line leaving the diamond. This is because that photon reached the maximum of six bounces, so no further rays were sent.

4.3 Many Photons

There is also the option to send many photons into the scene, the number of which is specified on the command line. When the user presses a key, all of the photons are recursively traced through the scene. When the photons are sent into the scene, data about the photons' paths is recorded. There is also a visualization created that the user can control with prespecified arguments. There are several options for the colors of the photon paths. These exist for one or many photons, but they only make sense to use with many photons because with a single photon path it is very clear what is happening in the scene. One option is to color each bounce path based on the number of bounces the photon has taken before it. For example, the first photon bounce in drawn in blue, the next is drawn in green, the next in yellow, the next in cyan, and then the next in orange. Then the colors repeat starting with magenta, which is the color used for the photons before their first bounce. Since we have limited our photons to six bounces, a color will never be used twice. This showed us that much of the light hitting the floor in certain areas was doing so on the same bounce count as the other photons. An example of this is shown in Figure 4.4. We used a hollow cylinder model to highlight this effect.

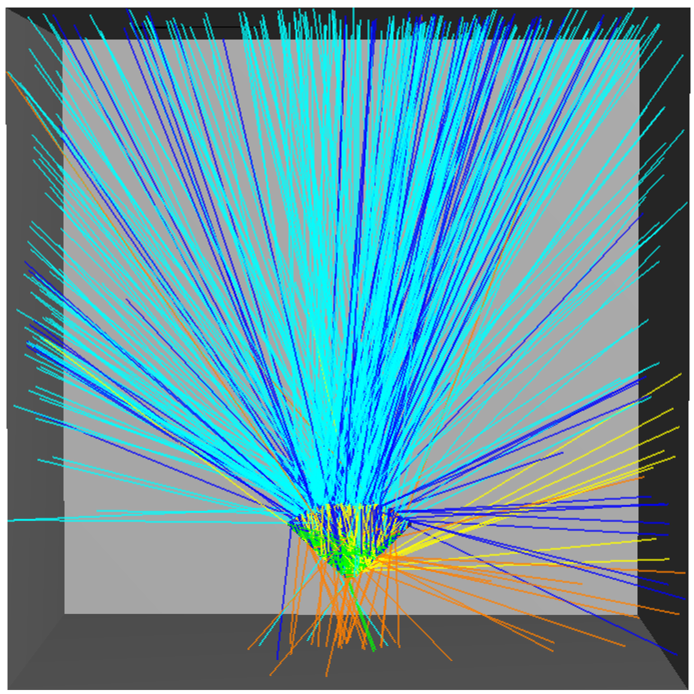

Figure 4.4: In this scene, the visualization to show the number of times a photon has bounced is shown. After the first bounce the path of the photon is shown in blue, then the colors green, yellow, cyan, orange, and magenta. In this image, we can see that there are concentrations of photons that have bounced a certain number of times. Here we can see that the photons that come out of the top of the diamond usually take a different number of bounces than the ones that come out of the bottom of the diamond.

4.4 Low Alpha Value

Another option is to change the alpha value of the photon paths, which is how transparent the paths are drawn. Using a low alpha value and a high number of photons creates a smooth image, showing where light is in the scene. Because the alpha value of the photon paths is lower, we can show a lot more photons in the same picture without the photons blocking each other as much. In a low alpha visualization, areas where there are more photons are a brighter white than areas that have a lower photon path density. Figure 4.5 shows an example of low alpha value scenes with white paths, and highlights how there is a lot of light coming out of the top of the diamond and how much the light bounces around inside the diamond. If we had not lowered the alpha value in this scene, most of the image would be white and it would be hard to tell where there were relatively more photon paths.

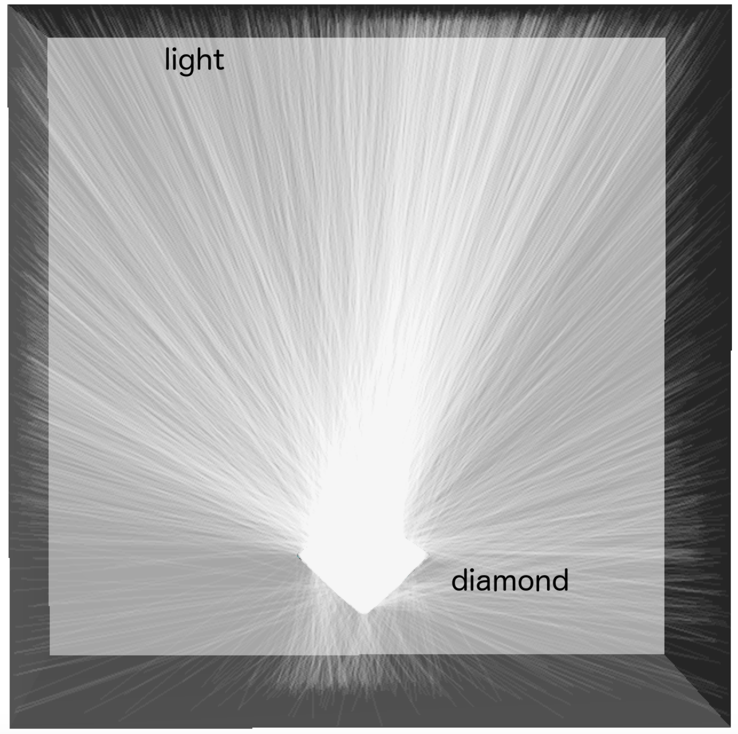

Figure 4.5: This scene has the photon paths of 50,000 photons drawn all in white with a very low alpha value, .05. From this image, it is clear how many of the photons come back out of the top part of the diamond because how bright the area above the diamond is. The brightness of the diamond itself shows how the photons have the tendency to bounce inside the diamond. There are more paths on the right because the photons coming from the light on the left don't have the first bounce drawn. By first bounce, we are referring to when the photons are coming from the light into the scene for the first time. Thus the photon path traveling from the light on the left to the diamond is not drawn.

4.5 Path Color

To see where different colors of light go in a scene, there is an option to change the photon path colors to be the colors of the photons based on their assigned wavelength values. In this visualization, each photon path has a uniform color for its entire path. Because of this, this method also helps follow a single photon throughout the scene.

4.6 First Bounce

One thing that originally made the visualization of many photons hard to interpret was the initial incoming path from the light, labeled as ’incoming light’ in Figure 3.1. Figure 4.6 shows what the visualization looked like with the first bounce included. This visualization is using the option that the paths will be colored by the number of times a photon has bounced. All of the first paths are purple. Because of the way the purple paths cover the scene, the first bounce was not included in any visualizations of many photons. It was, however, included in the single photon path case to show the initial location and angle the photon is coming from.

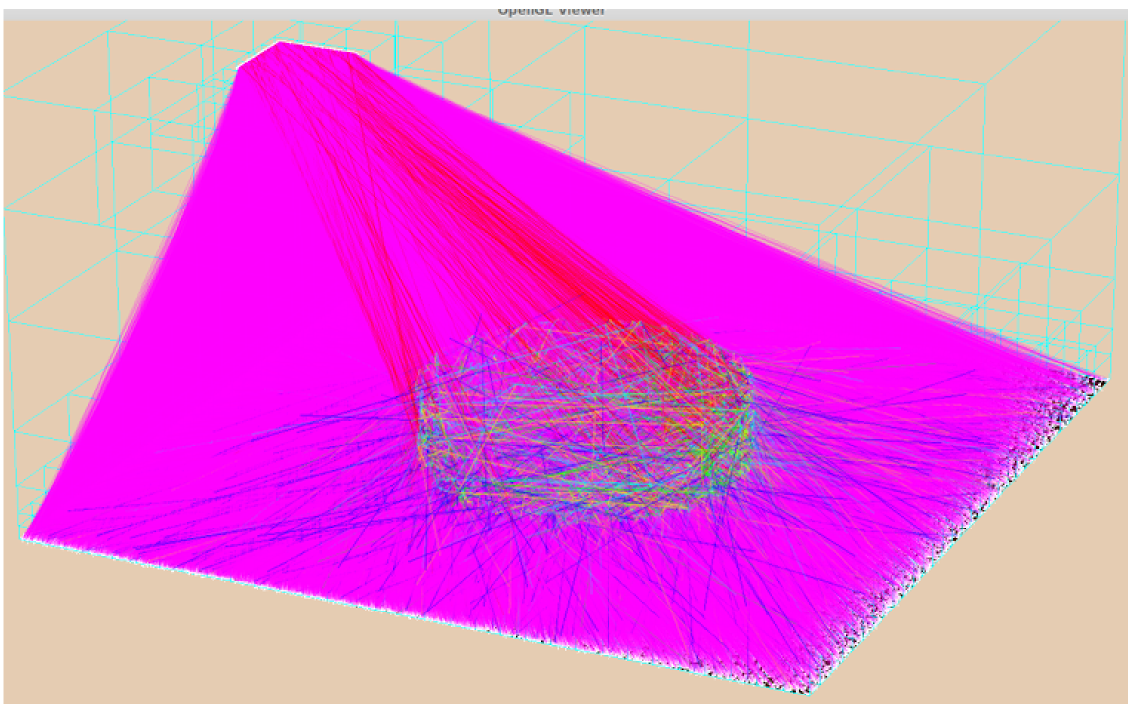

Figure 4.6: This figure includes the first path of the photon as it leaves the light. Because the photons are leaving the light in all directions and hitting all areas of the scene that face the light and aren’t occluded, these paths are very prominent. Because these first paths obscure the rest of the image, we do not visualize them in any of our other images.

4.7 Photorealistic Render

We also created a photorealistic visualization with code written by Andrew Faulds and Jesse Freitas. This code uses the photons sent into the scene with our photon mapping algorithm and finds what the RGB color value should be for each pixel. In the render shown in Figure 4.7, we can see the light reflecting off of the top bright faces of the diamond. We can also see the pattern of light as it refracts through the diamond. The caustic created by the diamond is visible beneath the diamond on the floor of the box. We can also see how the diamond is lighting up the top half of the box by sending light out of the top portion of the diamond.

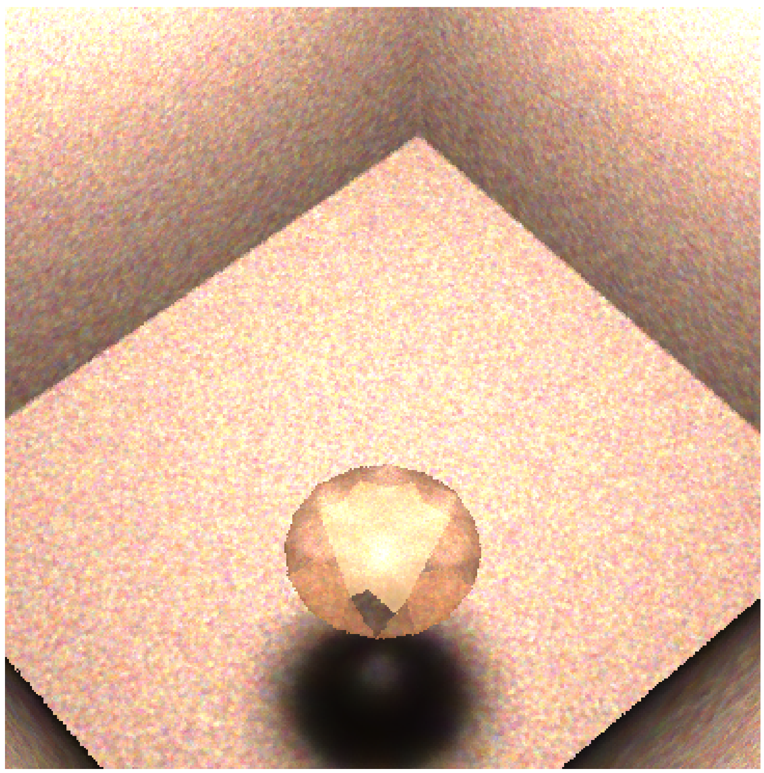

Figure 4.7: In this scene we sent 15,000,000 photons at the Eulitz diamond inside of a box. This particular render was created by looking at the nearest 100 photons to each pixel. The code to make this rendering was written by Jesse Freitas and Andrew Faulds.

4.8 Summary

We used visualizations to both understand how our code was working and to understand the data we were collecting. We drew small lines along the normals of the faces where the photons landed, which helped show the distribution of photons. We created options to draw the photons paths by the color of the photon, or by the number of bounces the photon had taken. We drew black lines along a photon's path when it escaped the scene to make it more clear where the photons were going. We also created visualizations of single photons going through the scene. When shooting many photons, we found that when We used a low alpha value we could see where the density of photon paths was highest where the paths looked brightest. We also found that the first path of the photons when they are coming from the light shouldn't be drawn because those paths blocked the rest of the image and didn't provide new information. In the next chapter we will discuss the tests that we ran to collect data and the results of those tests.

5. Testing and Results

We used four different diamond models for our tests, described in Section 5.1. For these tests, we used a single light to light the models in 3D space. We found that amount of light sent through the top of the diamond model varied based on the type of light being used in the scene.

5.1 The Models

To test the diamonds' cuts, we used 3D models made by Brian Truhlar. We also used one approximate sphere that we generated from subdividing the surfaces of a cube. This approximate sphere model was not chosen to replicate what a sphere would do, but to be a model that does not have any bias as to where it will send its photons due to its asymmetric shape. These models are bounded by a two-unit sphere. We used two round brilliant diamonds, the Eulitz Brilliant and the Ideal Brilliant, and a Princess Cut diamond, which has a square shaped top. All of the diamond models are shown in Figure 5.1. The round brilliant diamonds are the traditional looking diamond cut, and are accepted as the best type of cut for sending light out of the top of a diamond. The princess cut looks similar apart from having the square shape to the top. It is expected to perform more poorly than the round brilliants, but better than the sphere-shaped diamond. Apart from this ordering, we did not have a better intuition of how well each diamond model would do. All of these models are bounded by a two-unit cube.

All models are represented by a triangle mesh of their outer surfaces. We are modeling reflection and refraction, which both happen as light enters or leaves a medium. When light is entering a medium from air, it will hit the front face of a triangle in the object. When it is leaving the object and going back into air, it will hit the back face of one of the triangles in the object.

By using these models, we show that the round brilliant cuts perform the best mathematically. Between the two, we see that the small changes in angles between faces actually do make a difference in the way light leaves the diamond. The princess cut diamond shows a significant performance difference from the round brilliant. We used the sphere model like a control group to show what would happen if no effort went into designing a shape that would send light out of the top of the model.

Figure 5.1: These are the diamond models that we used in this simulation. From left to right this is the sphere model, the Princess Cut model, the Eulitz model, and the Ideal model. The Eulitz model and the Ideal model are round brilliant diamonds. It can be seen that the Princess model has a square shaped top. The round brilliant diamonds are the most similar, and both have 58 faces. This is not the quad count of the models we are using, but the number of faces on the actual diamond. They differ in the angles between the different planes of the faces. The number of quads in each model are 144, 76, 128, and 144 from left to right respectively. There are coplanar faces because some of the faces on a diamond cannot be mapped to a single quad.

5.2 Initial Testing

5.2.1 Running Environment

We put the models in open space. We have also tested diamond models inside of a box for clarity, but since the walls of the box in our images absorb all light that hits them and do not occlude the diamond, the box walls do not have any effect on the photon bounces and were not included in our final simulations. We sent 20,000,000 photons from the light source and allowed each of them to bounce up to 6 times. Each photon was randomly assigned a wavelength from the visible spectrum between 380 and 780. This does not accurately reflect what would happen in a real situation because lights and the sun do not have an even wavelength distribution such as this. Photons come from a random location from the surface of the light.

Figure 5.2 shows this setup.

Figure 5.2: This image shows the environment in which we did the tests. The light is located 8 units above the center of the diamond and points downward at the diamond. In this image, the light is directly overhead the diamond, as it is in Section 5.2.3.1.

5.2.2 Directed Light

One of the tests we ran used a directed light source from overhead but slightly off-center and directed the light toward the center of the diamond. Because this is a directed light source, that means all of the photons coming out of it go in a single direction. We used the direction from the center of the light source to the center of the diamond. The light source is a two unit square, from -3.5 to -0.5 units in the x and z directions and 8 units above the center of the diamond. This is visually very similar to the setup in Figure 5.2, but the light is slightly off enter.

The diamond shape aims to have photons leaving the faces that point upward. To see which diamond shape did this the best, we compared which faces the most photons are leaving. We compared the angle of the faces' normals with vertical (0, 1, 0), and grouped them into four categories depending on the angle between their normal and straight up: less than 45, between 45 and 90, between 90 and 135, and greater than 135. Diagram 5.4 shows a face normal and its angle with up. Figure 5.3 visualizes which faces are in each of those categories with the colors, blue, orange, grey, and yellow respectively.

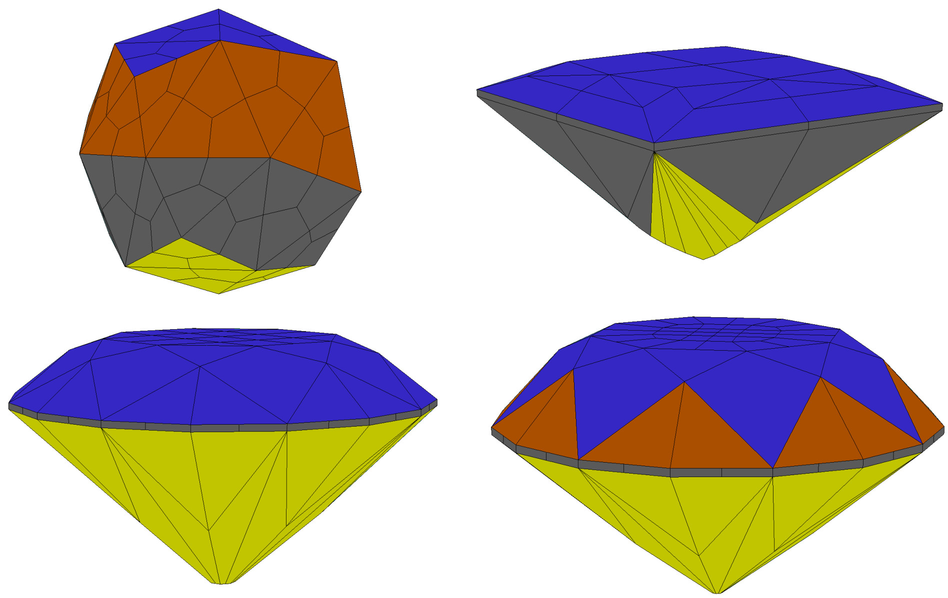

Figure 5.3: This image shows each of the diamond models with their faces colored according to the normal of the face. In the top left corner is the sphere approximation, in the top right the Princess cut, in the bottom left the Eulitz cut, and in the bottom right the Ideal cut. The faces are colored blue, orange, grey, and yellow to signify the angles between the face normals and a vector pointed straight up. These groups are less than 45 degrees, between 45 and 90 degrees, between 90 and 135 degrees, and greater than 135 degrees, respectively. As we can see from this image, the Eulitz cut does not have any faces that are in the 45 to 90 degree category. There are faces in that range on the Ideal cut, but all of the corresponding faces on the Eulitz cut fall into the less than 45 degree range.

In Figure 5.5 we compare these angle statistics for the four diamond shapes. We can see that the two round brilliant diamonds send the most light out of the top with the Eulitz brilliant sending 6.51% of light out of the top 45 degrees and the ideal brilliant sending 5.84% out of the top 45 degrees. The princess cut diamond had the next most coming out of the top 45 degrees with 4.62%, and lastly the sphere had 2.91%. This is what we expected since the round brilliants were designed to send as much light as possible out of the top. In contrast, the design of the square cut diamond isn't mathematically optimized for sending light out the top. In Figure 5.5 we can see that the Eulitz diamond is sending most light out of the top 45 degrees. If looking at a wider angle than 45 degrees however, the sphere and the ideal diamond both send more light out of the top 90 degrees.

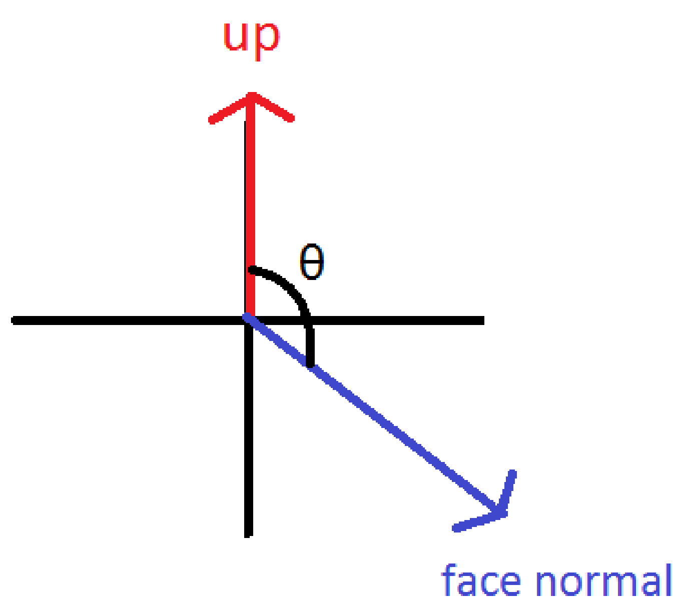

Figure 5.4: For our measurements, we looked at how many photons left each face, and grouped the faces together by their normal vectors. This diagram shows the angle between a face normal vector and up.

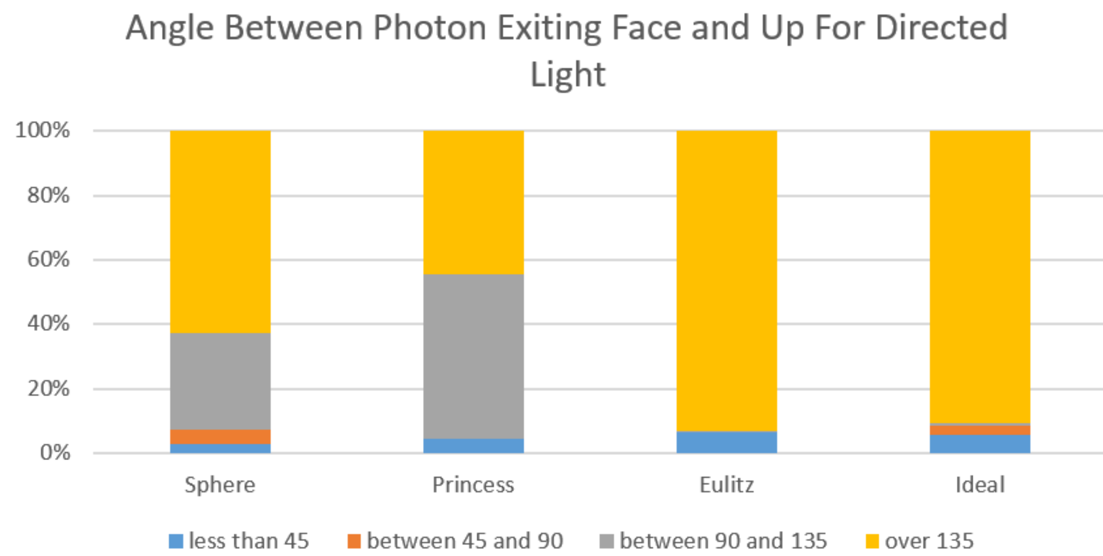

Figure 5.5: This graph shows the angle of the faces that the photons are leaving. Each face's normal was compared with 'up' and placed into the four categories of less than 45 degrees, between 45 and 90 degrees, between 90 and 135 degrees, and over 135 degrees. We have used percentages, so the scale is from 0 to 100. This chart shows that most of the light went through the bottom part of the diamonds. Of the light that came out of the top 45 degrees of the diamond, we can see the two round brilliant diamonds have the most light coming out the top, followed by the princess diamond. The light that comes out of the top of the diamond can be explained by the path of the light shown in Figure 3.1. The light in these scenarios is coming from above the light as it is in that figure.

Figure 5.6 shows the angle at which photons are leaving each diamond model. From this graph we can see that there is a large spike of photons leaving the diamonds around 160 degrees. This is because of the direction that the photons are shot at the diamond and because diamonds are highly refractive. We had expected that the diamond shape models would have more photons with a smaller angle from the top, but this was not the case. Upon inspection of visualizations of photon paths for this test, we see that many photons are refracting into the top of the diamond, and then refracting again out the bottom faces of the diamond. In section 5.3.1, we will explore other positions of directed light and show that this result was atypical. This result was due to this particular location of this directed light.

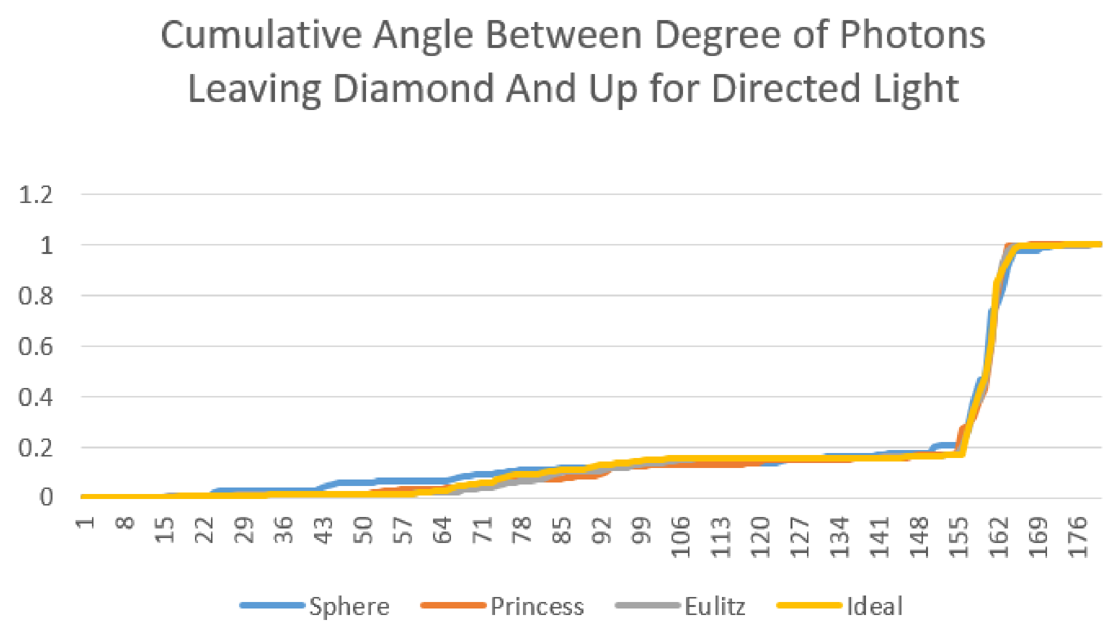

Figure 5.6: This graph shows the degrees at which the photons are leaving the diamond. The graph is cumulative, so the value at a certain degree represents the percentage of photons that have left the diamond and a certain angle from the top more towards the up. Ideally, we would see that more photons come out at angles closer to straight up, so the ideal shape for a diamond on this graph is to rise very quickly and reach nearly 100% by 90 degrees. This graph shows that using a directed light, the shape of the diamond does not matter as much. All of the diamond shapes behaved in relatively the same way, creating photons leaving at very similar angles.

5.2.3 Undirected Light

To test the system with undirected light, we used the same size and location of light and diamond as we did for directed light. With the undirected light source, we allowed photons to come out of the light source in random directions, although we did not allow them to come out of the back of the light. We sent 20,000,000 photons from the light in the same location, but allowed the photons to go in any direction from the frontward facing portion of the light. This is similar to an indoor setting with a light on in a room. Because photons that don't hit the diamond on their first bounce are not counted, we are only looking at photons that come from the light in the direction of the diamond. In this simulation, we did not resend photons that did not hit the diamond directly from the light.

In Figure 5.7 we can see that the three diamond shapes have most of their photons coming out of the faces facing the top 45 degrees. The Eulitz diamond has the highest percentage of photons coming out of the top 45 degrees with 85.95%. That is closely followed by the ideal brilliant with 75.91%. The princess diamond had 68.52%, which is lower than the round brilliants, but much higher than the 11.69% of the sphere. One thing we see in both directed and undirected light is that the ideal brilliant sends many more photons in the between 45 and 90 degree category. There are no photons leaving faces between 45 and 90 degrees on the Eulitz, on the ideal, 15.39% of the photons leave in the 45 to 90 angle range. Adding the 0-45 degree range and the 45-90 degree range for the ideal makes 91.30%, which is more than the Eulitz brilliant.

Unlike the directed light, we see more of a spread of the exit angle of the photons. Figure 5.6 shows the angle at which the photons left the diamond. Here we can see that the ideal diamond performs the best at sending the photons up, but it is closely followed by the Eulitz brilliant. Like in the diagram of the angle of the faces the photons are leaving, in this case the princess diamond is near the two round brilliants in, but lower. The sphere sent a significantly lower percentage of photons up, which shows the benefit of the diamond shapes.

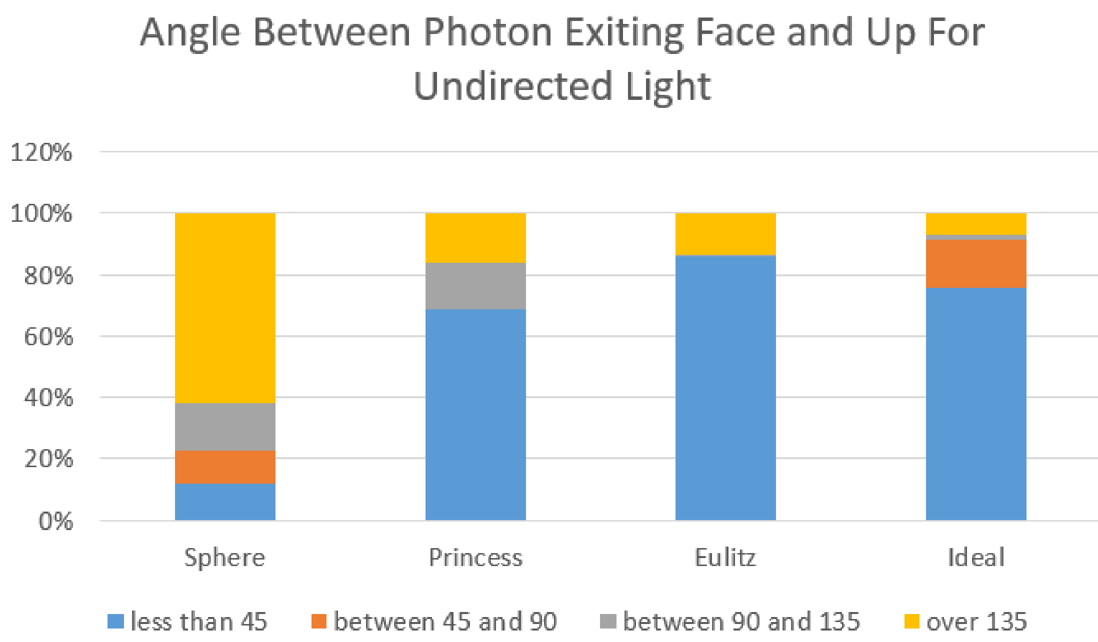

Figure 5.7: This graph shows the angle of the faces that the photons are leaving, the same as Figure 5.5. The three diamonds show that the majority of their light come out of the top 45 degree faces.

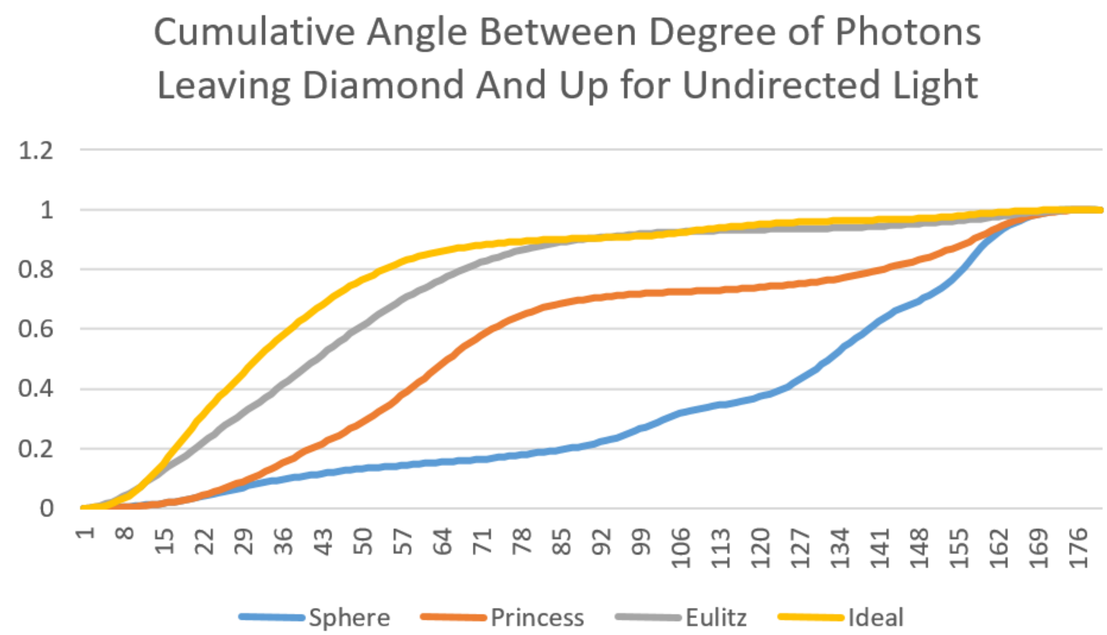

Figure 5.8: This graph shows the angle between the direction photons left the diamond and straight up, as in Figure 5.6. Unlike in Figure 5.6, in this graph we can really see how the diamonds perform better than the sphere in sending their photons upward. This difference is because we are using undirected light in this test. The two round brilliants also do better at this than the sphere, as expected. In this case, the ideal brilliant sent more light closer to up than the Eulitz brilliant.

5.2.3.1 Light Directly Overhead

We also ran the undirected light test with the light directly overhead the diamond, from -1.5 to 1.5 in the x and z direction, still 8 units above the center of the diamond. This setup is shown in Figure 5.2.

In Figure 5.9, we can see that the results for putting the light directly overhead are similar than an off-center light. 90.04% of the photons leave the Eulitz diamond's top 45 degree faces when the light is directly overhead as opposed to 85.95%, and 85.78% leave the ideal's top 45 degrees, as opposed to 75.91%. There are fewer photons leaving the 45 degree to 90 degree range for the ideal brilliant with the light overhead, so the increase in the number leaving the ideal's top 90 degrees is actually only 1.53%.

There was more of a change with the sphere. The sphere goes from having 22.47% of the photons coming out of the top 90 degrees with the light slightly off centered to 34.93% with a centered overhead light. In Figure 5.10 we can also see that the sphere has less photons coming out at an angle very close to down than in Figure 5.8. With the light directly overhead and with the very high index of refraction of diamond, the light seems to be shining right through the sphere significantly more. The way the sphere does worse with sending light out of the top of the model when the light is directly overhead highlights how well the traditional diamond cuts send light out of the top when light comes in from directly overhead, illustrated in Figure 3.1.

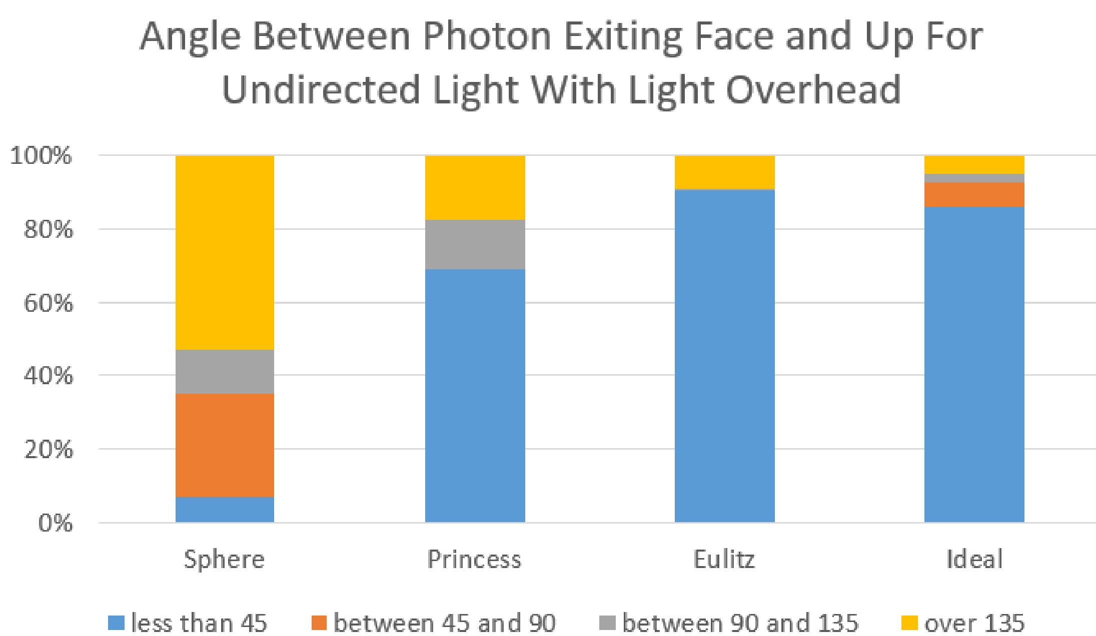

Figure 5.9: In this figure, we can see that the results are very similar when the light is directly overhead. For the two round brilliant diamonds, just slightly more light is coming out of the top 45 degrees.

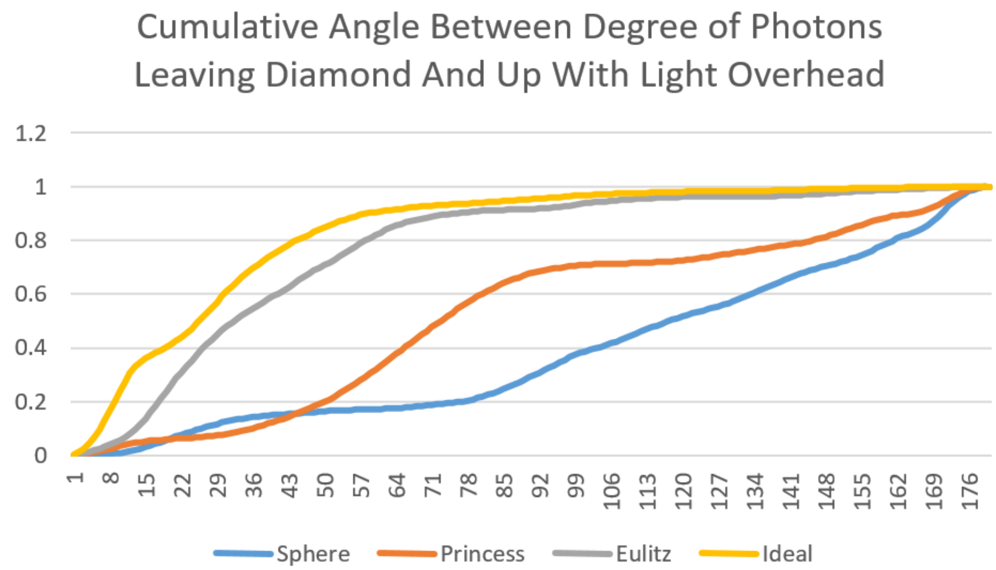

Figure 5.10: Here we can see that especially in the ideal diamond, the photons come out at an angle closer to up. The sphere also has fewer photons that are leaving very close to down.

5.3 Rotating the Light Around the Diamond

Another test that we ran was to rotate the light around the diamond. For this set of tests, we used only the Ideal diamond cut, which we found to be the most effective at sending light out the top in section 5.2. We took the light starting directly above the diamond and rotated it around the diamond in 30 degree increments. The different positions of the light can all be seen in Figure 5.11. For these tests, we fired as many photons as necessary to attain 10,000,000 photons that hit the diamond. Since the light was coming at the diamond from different angles, we wanted to ensure that each simulation had the same number of photons hitting the diamond.

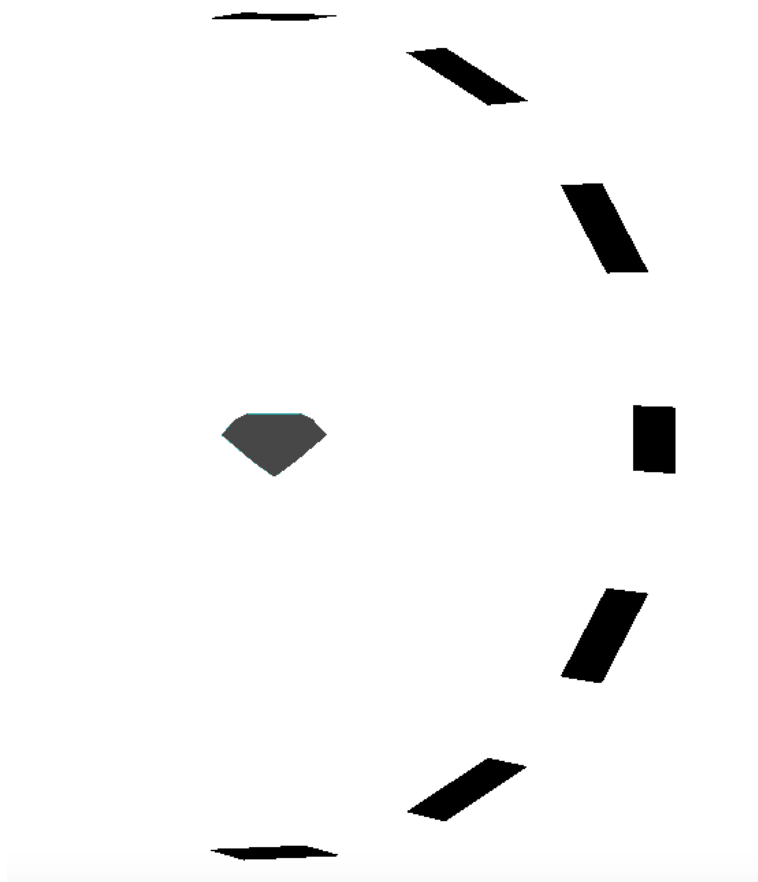

Figure 5.11: This image shows all of the positions of the light as it is rotated around the diamond for different simulation runs. Although all of the light sources appear in this image, each simulation had only one light. The light is rotated in 30 degree intervals, starting with 0 degrees and ending with 180.

5.3.1 Directed Light

When sending directed light at the diamond, we based the size of the light off of the diamond. Because all rays were sent in the same direction, a photon starting at a particular spot on the light could only hit one spot on the diamond. Any area of the light that could not hit the diamond was wasteful because any photons that started from these areas would count toward the 10,000,000 needed for the simulation. We therefore set the dimensions of the square light such that it was big enough that all portions of the light that could hit the diamond were present. We used a two unit by two unit square. The distance from the light to the diamond also did not affect the result because the photons would hit the diamond at the same angle in the same spot regardless of how far away the light was. We limited our testing to seven different rotations of light around the diamond because of the time it took to run each simulation.

For this test, we recorded how many of the photons left each of the faces of the diamond. We grouped them based on the normal vectors of the faces as shown in Figure 5.3. We then compared how many left faces whose normal vectors were within 90 degrees from up, as shown in the Figure 5.12.

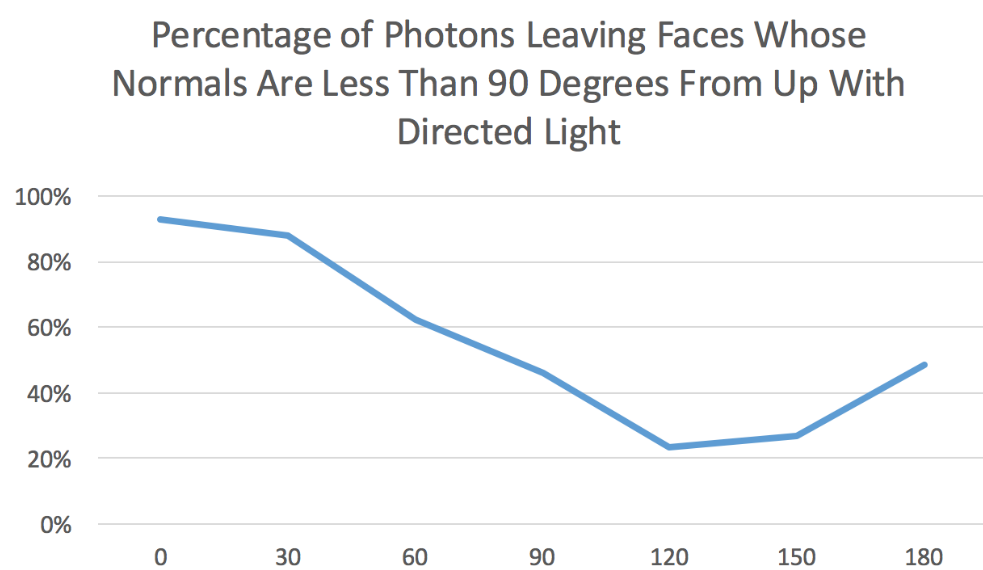

Figure 5.12: This graph shows the results of an experiment where light was shot directly at a diamond from a light that was rotated around the diamond. The percentage of photons that left through faces whose normal vector was less than 90 degrees is plotted.

We can see that the angle that the light hits the diamond greatly impacts how much light comes out the top of the diamond. When the light was directly above the diamond, 92.71% of the light came out the faces whose normals are less than 90 degrees from up. We see that this number decreases as the light is rotated around the diamond. We see that this trend continues until the light reaches 150 degrees, where the amount of light coming out the top portion of the diamond increases from 23.27% at 120 degrees to 27.02%. These results suggest that the very few amount of photons coming out of the top of the diamond with directed light in Section 5.2.2 was due to what happened to be a poor angle and lighting placement.

5.3.2 Undirected Light

Unlike a directed light source, when using an undirected light source, the size and the distance of the light from the diamond affect the angle at which the photons can hit the diamond. A light source of the same size but closer to the diamond allows a photon to hit the diamond at shallower angle. By shallower, we mean an angle further from the normal angle of the face. This is because the photon has less distance to travel down toward the diamond, but the same amount of distance it can travel from one side of the light source to the other side of the diamond. For the same reason, a light that is moved farther away but is scaled to be larger will send photons toward the diamond in the same way as a smaller, closer light. We chose a light source that modeled a realistic situation. The light source represents a two-foot square light that is seven feet about the diamond.

Because an undirected light sends light in all directions, the percentage of photons that hit the diamond was much lower than when we used the directed light. To send enough photons into the scene that 10,000,000 of them hit the diamond, many more must be sent into the scene. The intersection calculations of these photons that never hit the diamond take up a significant amount of time, even though they are not counted toward the 10,000,000 photons for the simulation. We considered several ways of speeding up the program to have fewer photons that miss the diamond when they are sent into the scene from the light.

One method we considered was to choose the direction of the photon by choosing a point on a bounding sphere of the diamond to shoot at. This would significantly increase the percentage of photons that hit the diamond on the first try because many of the directions that would miss the diamond would no longer be possible. Upon further inspection, we realized that this method would not uniformly distribute the photons from the light source. From the point of view of a photon being shot from the light, the direction where the sphere is thickest will have a higher chance of being selected than other directions. Another method we considered was placing a plane in front of the diamond and choosing a random point on that plane for each photon to shoot the photon towards. This method also does not get an equal distribution of photons throughout the portion of the hemisphere that contains the diamond because the plane is flat, unlike the curved hemisphere.

In our solution, we uniformly choose a random direction in the entire hemisphere. We then use a square between the diamond and the light source as a filter. This square completely eclipses the diamond from the point of view of the light. Before being tested for intersection with the diamond, a photon's ray is tested for intersection with the square. If the ray for the photon does not intersect with that face, we do not bother to test to see if the photon will intersect with the diamond model, because we know that it will not. Since there are over 200 quads in the diamond, we are able to save a significant amount of time by avoiding these comparison tests. Even so, it still took about five times as long to run the undirected simulation as it did to run the directed one. The results are shown in Figure 5.13.

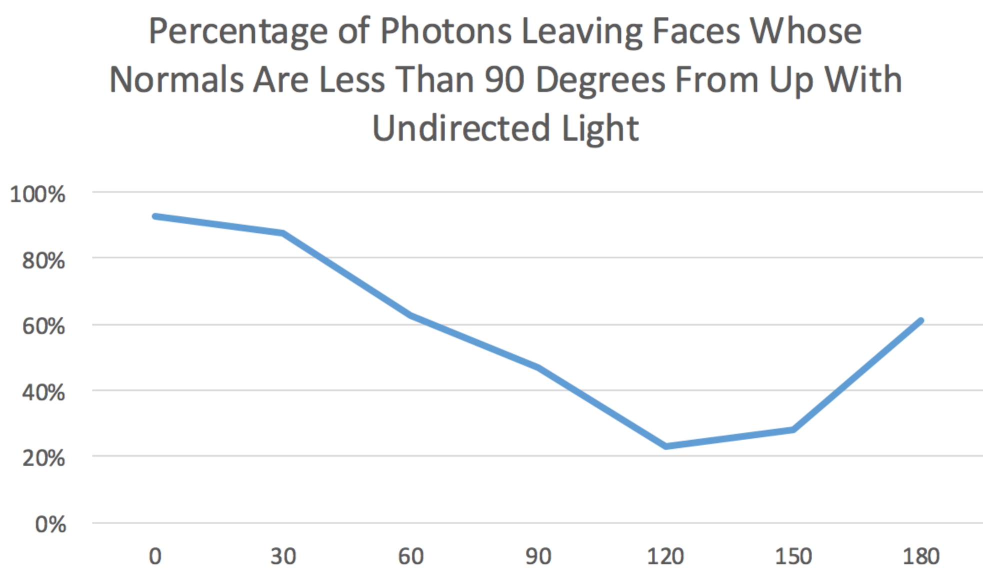

Figure 5.13: This graph shows the results of an experiment where light was shot from an undirected light at a diamond as it was rotated around the diamond. The percentage of photons that left through faces whose normal vector was less than 90 degrees from up is plotted.

Using the undirected light, we saw the same general trend as with the directed light. The angles very close to overhead had the most light coming out of the top portion of the diamond with the angles between 90 and 180 sending the least light out the top. In fact, other than the 180 degree angle, the amount of photons sent out the top portion of the diamond when lit with an undirected light source was within a single percent of when a directed light source was used. When the light source was directly below the diamond, 61.10% of the photons came out of the top portion of the diamond as opposed to 48.65% with a directed light source. This suggests that the straight up angle is not a good angle to send photons at the diamond if trying to get them to come out the top portion of the diamond.

5.4 Summary

We used four different diamond models in our test: two round brilliant models, a princess cut model, and a sphere approximation shape. We shot photons at the diamond models from a single light source in open space. We used both direct and undirected light in our tests. In our initial test that used directed light, we found that most of the light went straight through the diamond and out the bottom portion of the diamond. Many of the photons left the diamond at about a 160 degree angle from up. When we used undirected light, many more of the photons came out of the top part of the diamond. More of the photons came out of the faces facing less than 45 degrees to up of the Eulitz diamond than the Ideal diamond, but the Ideal diamond had more photons leave at angles closer to up. Both of these round brilliant models performed better than the Princess cut diamond in both measurements. All of the diamond cut shapes performed better than the sphere approximation shape in both measurements as well. We also found that diamonds send most light out of the top when lit from directly above, and least light out of the top when lit from approximately 120 degrees from the top. In the next chapter we will discuss our conclusions from the data we collected when running our simulation.

6. CONCLUSION AND FUTURE WORK

6.1 Conclusion

We have clearly shown in this study that the round brilliant diamonds that we used perform much better at sending photons back out through the top of the diamond when sending light in from the top, when compared to a sphere shape and a princess cut diamond. Although the Eulitz diamond tends to send more photons out in the top 45 degrees than the ideal brilliant, the ideal brilliant sends more photons in the 45-90 degree range, and thus sends more photons in general out of the top 90 degrees. It also sends more photons out with angles closer to up. For these reasons, of the diamonds we tested, we believe that the ideal brilliant diamond is the best diamond shape for sending light out of the top of a diamond. We also saw that the princess cut diamond behaved significantly better than a sphere, but nowhere near as well as either of the two round brilliant diamonds.

We have also shown that diamonds send more light out the top when they are lit with an undirected light than when lit with directed light in the case of the slight offset overhead light. The directed light mostly refracted through the diamond and came out the bottom of the diamond when the diamond was lit from the top. It left little coming back out of the top. We can also see that the position of the light is better directly overhead than slightly offset, but there is not significant improvement from a slightly off-centered light.

When rotating the light around the diamond from directly above to directly below, we saw that the best position for the light was directly above. We saw that the worst positions of the light were at 120 degrees and then 150 degrees respectively. We also saw very little difference between using an undirected vs a directed light source, except for at the position directly below the diamond. Here we saw the diamond sent most photons out of the top of the diamond when it was lit with an undirected light source.

6.2 Limitations and Future Work

The physics in this system isn't exactly true to real physics. It wouldn't be feasible to model the extremely complex way light actually behaves. This includes both the number of photons we are using, and properties of light such as change in wavelength and polarization. Polarization is when the oscillation of all of the light waves line up in the same orientation. In unpolarized light, like the light we are modeling, the oscillation can be in any direction. There are also limitations in the system that pertain to the way it reads in and processes its models.

The system currently only accepts quad meshes. There is currently a method of changing triangles in a mesh to quads, but it uses Catmull-Clark [11] which triples the amount of faces in the diamond mesh. A way to fix this limitation would be to extend the mesh class to have both quad and triangle faces. Many of the functions would have to be reimplemented to allow for both quads and triangle faces, such as looping around a face. The way the mesh is implemented now, traveling around a single face via the face's edges can be done with a fixed number of edge traversals. This limitation does not actually impact the functionality of the system or the results, but being able to use meshes with triangles would be a more elegant solution. Also, we would not have to subdivide triangle faces, which would lead to smaller meshes and faster running times.

Another way to accept triangle shapes in the quad mesh that is currently implemented is to allow for three of the points in the mesh to be co-linear. Having collinear points affects the code to find the normal of a face, so that code had to be changed.

Another limitation is that our photons never change wavelength. In the real world, when a photon enters a medium, its wavelength changes. This change in wavelength is relatively small and since the change in angle from wavelength is already small with a large wavelength change, we have decided not to implement this change in wavelength. This is something that we could do in future work. I implemented wavelength because before doing it, I did not realize how small of a change the wavelength had on the angle of refraction.

We are running 20,000,000 photons through our system, which is a substantial number, but it is still not accurate to real life. The number of photons that leave a light in a second would be many orders of magnitude higher than the number that we are sending into the scene. A 60 watt light bulb emits about 3.3×10^{17} photons per second [12]. Of these, the only ones that would be seen in an image are the ones whose paths end at the cameras lens. Because this is so inefficient to do, we use reverse ray tracing from the lens into the scene for the first bounce of light (from the camera) and use photon mapping for all other bounces of light coming from the lights direction. This does not affect the conclusions that we have drawn. The photons leaving the diamond would do so even if we did not do the reverse ray tracing step. Also, in a room where walls are probably diffuse, light would bounce randomly from the wall and that light would be just as likely to go to the camera lens as anywhere else. While we could not actually simulate the number of photons in a diamond in reality, we could simulate more of them in the future with better resources.

In reality, when light reflects off of a surface like a diamond, it becomes partially polarized. This change in polarization affects when light is absorbed. In our system we always assume that photons are not polarized, so when calculating absorbency, we take the average of the vertical and horizontal absorbency values. To extend our system to keep track of polarization, we would make the polarization of the photon a property of the photon class and would have to update it at every bounce.

For future work, we would like to do a user study to see what diamond cuts people prefer. We would give them the same diamond models we used, in a more true to life diamond size. Making sure that the diamonds are oriented face up, we would ask subjects to rank how attractive they found each diamond, and also how sparkly they thought each diamond was. We would make sure to be in a room with a single undirected light source, similar to our simulation, and then see how our results match those responses.

Another thing we would like to do in the future is to test more shapes of diamonds. We got our diamond models from Brian Truhlar, which he modeled out to be the correct specifications for each type of diamond that we used in our simulation. We could not find these models already in existence.

For another item of future work, we would also like to use statistical testing methods to determine the certainty of our results. We have a high sample count, but we would like to determine the confidence value of our results. This would be done via statistical analysis of our results, taking sample size into account.

REFERENCES

[1] M. Tolkowsky, *Diamond Design: A Study of the Reflection and Refraction of light in a Diamond*, 1st ed. New York, NY: Spon and Chamberlain, 1919.

[2] H. W. Jensen, "Global Illumination Using Photon Maps," in *Proceedings of the Eurographics Workshop on Rendering Techniques '96*. London, UK: Springer-Verlag, 1996, pp. 21–30.

[3] C. M. Goral, K. E. Torrance, D. P. Greenberg, and B. Battaile, "Modeling the Interaction of Light Between Diffuse Surfaces," *SIGGRAPH Comput. Graph.*, vol. 18, no. 3, pp. 213–222, Jan. 1984.

[4] T. Whitted, "An Improved Illumination Model for Shaded Display," *Commun. ACM*, vol. 23, no. 6, pp. 343–349, Jun. 1980.

[5] S. Guy and C. Soler, "Graphics Gems Revisited: Fast and Physically-based Rendering of Gemstones," *ACM Trans. Graph.*, vol. 23, no. 3, pp. 231–238, Aug. 2004.

[6] GIA, "Diamond," New York, NY, 2002-2015, [Online]. Available: http://www.gia.edu/diamond-quality-factor (Accessed November 20, 2015).

[7] C. Wild and E. Worner, "The CVD Diamond Booklet," [Online]. Available: http://www.diamond-materials.com/downloads/cvd_diamond_booklet.pdf (Accessed March 27, 2016).

[8] G. Woan, *The Cambridge Handbook of Physics Formulas*, 1st ed. New York, NY: Cambridge University Press, 2000, Cambridge Books Online.

[9] J. Walker, *Fundamentals of Physics*, 9th ed. Hoboken, NJ: John Wiley and Sons, Inc., 2011.

[10] D. Greve, "Reflections and Refractions in Ray Tracing," November 2006, [Online]. Available: http://graphics.stanford.edu/courses/cs148-10-summer/docs/2006--degreve--reflection_refraction.pdf (Accessed April 1, 2016).

[11] E. Catmull and J. Clark, "Recursively Generated B-spline Surfaces on Arbitrary Topological Meshes," in *Seminal Graphics*, 1st ed. New York, NY: ACM, 1998, pp. 183–188.

[12] R. Boothe, *Perception of the Visual Environment*, 1st ed. New York, NY: Springer Science and Business Media, 2001.

[13] *Seminal Graphics*, 1st ed. New York, NY: ACM, 1998.