Supporting Increment and Decrement Operations in Balancing Networks

by

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Balancing Networks

Introduced by Aspnes, Herlihy and Shavit in STOC'91

Distributed data structures used for:

- shared counters
- barriers
- load balancing

Advantages:

- low contention
- non-blocking

Limitation: basic operation is increment (+1)

We would like them to support: decrement (-1)

Decrement is good for:

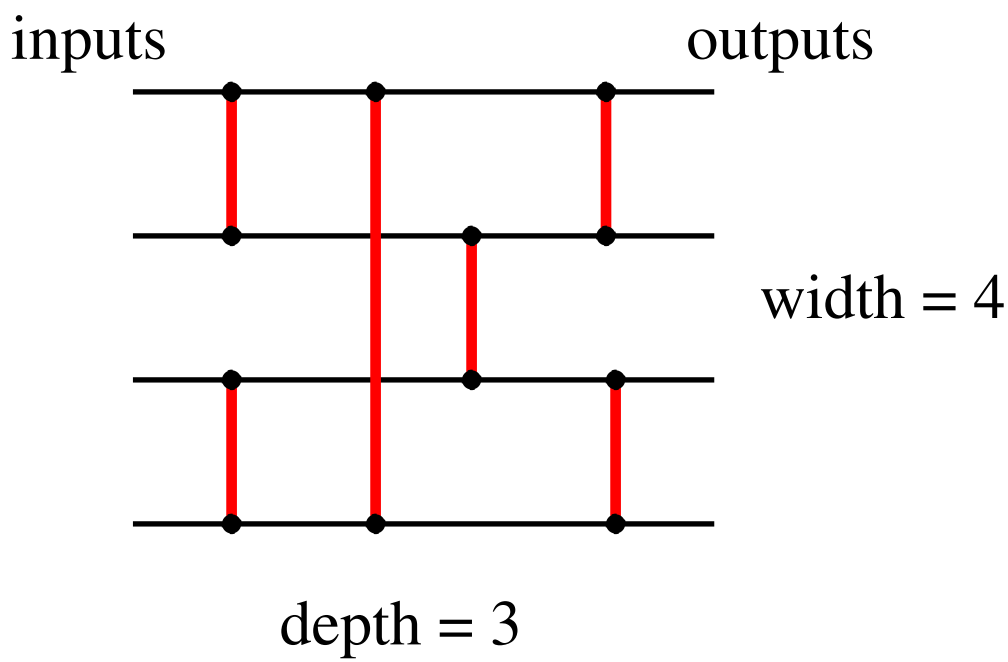
- concurrent “pools” and “stacks”
- semaphores

We show:

certain balancing networks can support both increments and decrements

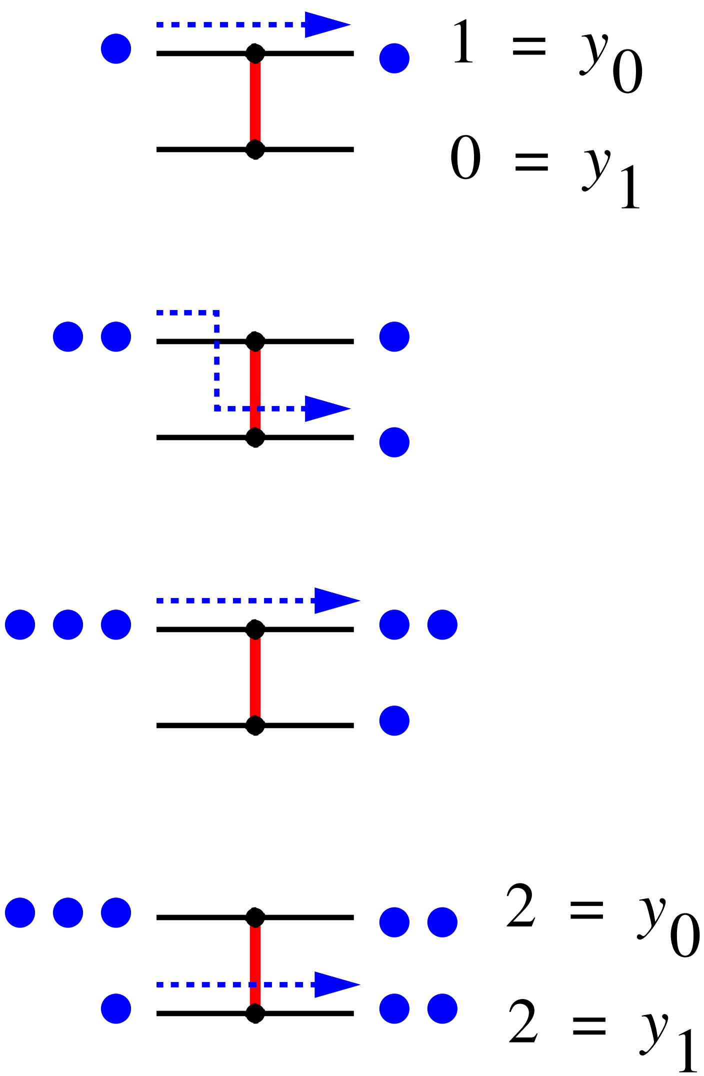
Balancing Networks in Detail

They look like sorting networks:



Constructed from: balancers

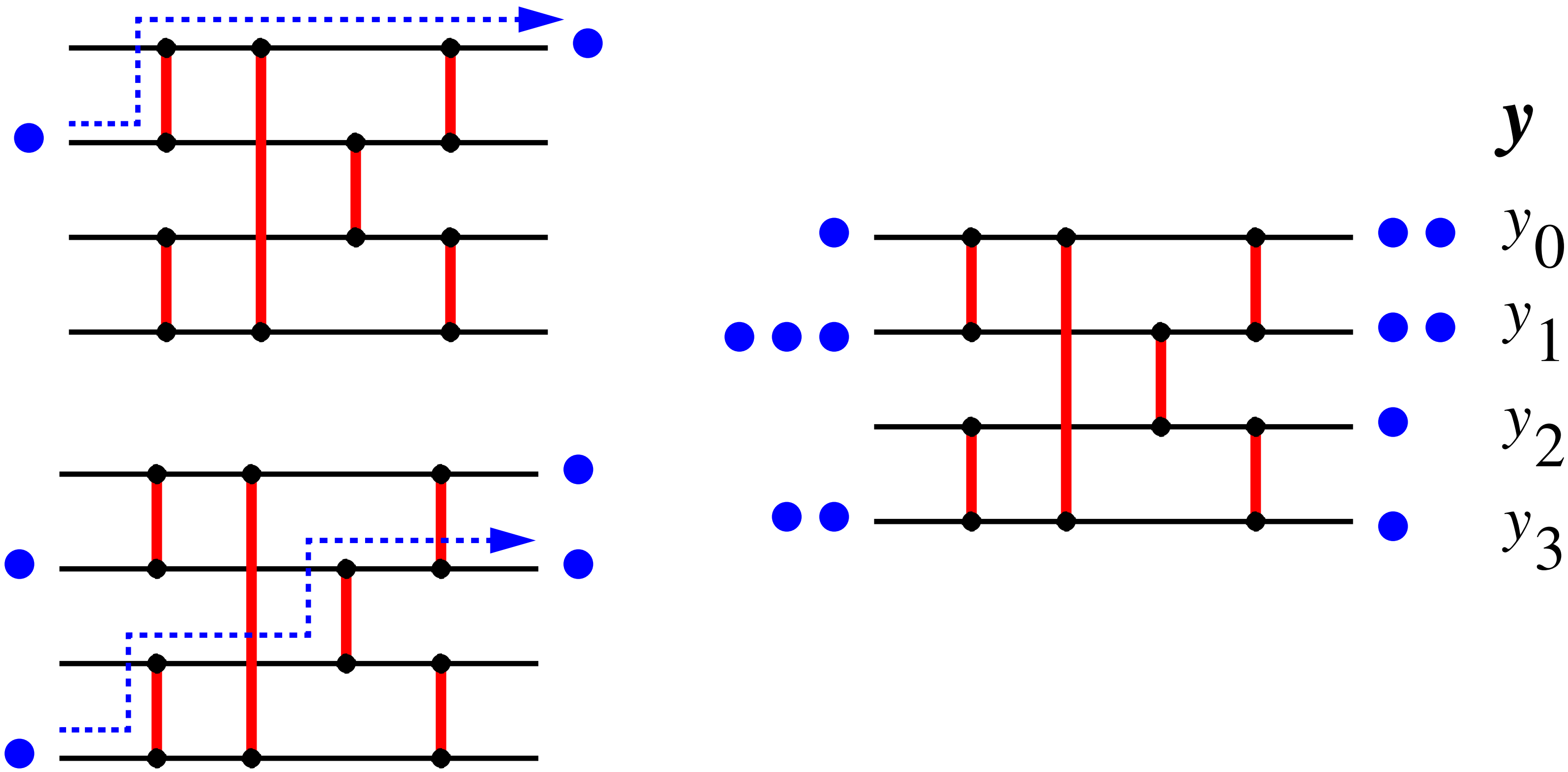
Balancer: routes tokens (increment +1)



Step Property:

$$0 \le y_0 - y_1 \le 1$$

Balancing network



Counting networks satisfy the step property:

on output vector ***y***, $0 \le y_i - y_j \le 1$ for $i < j$

K-smoothing networks satisfy the K-smoothing property:

on output vector $\boldsymbol{y}$, $\left|y_i - y_j\right| \leq K$ for all i, j

A boundedness property is:

- a subset of the K-smoothing property
- closed under addition with a constant vector

Examples of boundedess properties:

- step property
- K-smoothing property (trivially)

Supporting Decrements

For increment (+1) we used: tokens

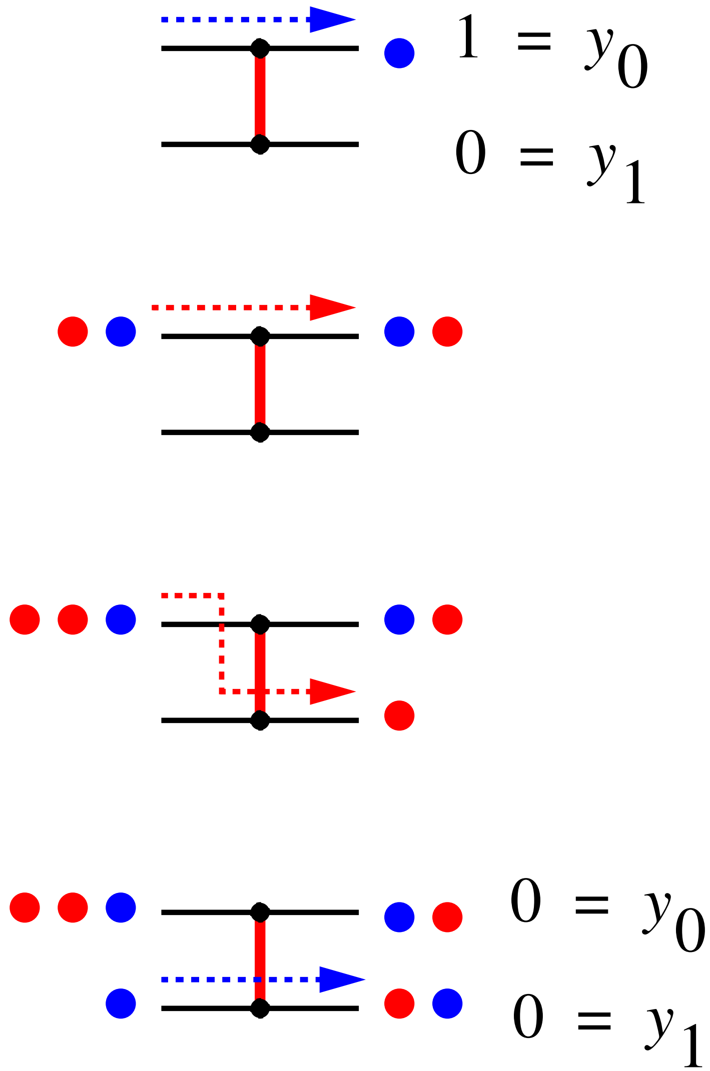
For decrement (-1) we will use: antitokens

Antitokens cancel the effect of the tokens:

Token + Antitoken = 0 (elimination)

Antitokens were introduced by Shavit and Touitou in SPAA'95

Tokens (+1) and antitokens (-1) in a balancer



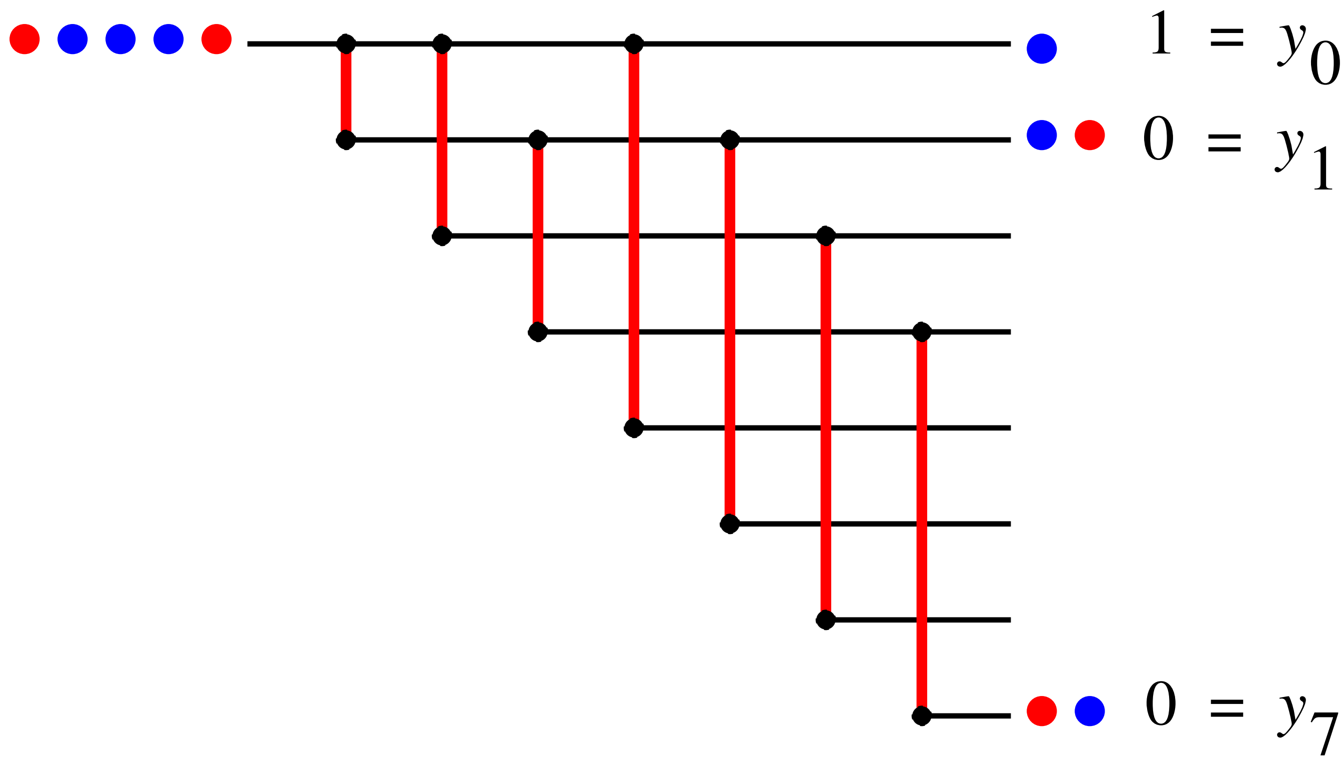
Step Property:

$$0 \le y_0 - y_1 \le 1$$

Shavit and Touitou showed:

counting trees (a kind of counting networks)

support decrements



Step property still holds: $0 \le y_i - y_j \le 1$ for $i < j$

We show a more general result:

if a balancing network satisfies a boundedness property with increments then

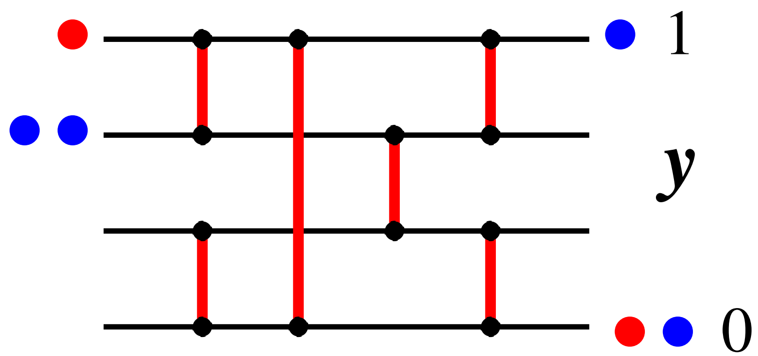
it also supports decrements

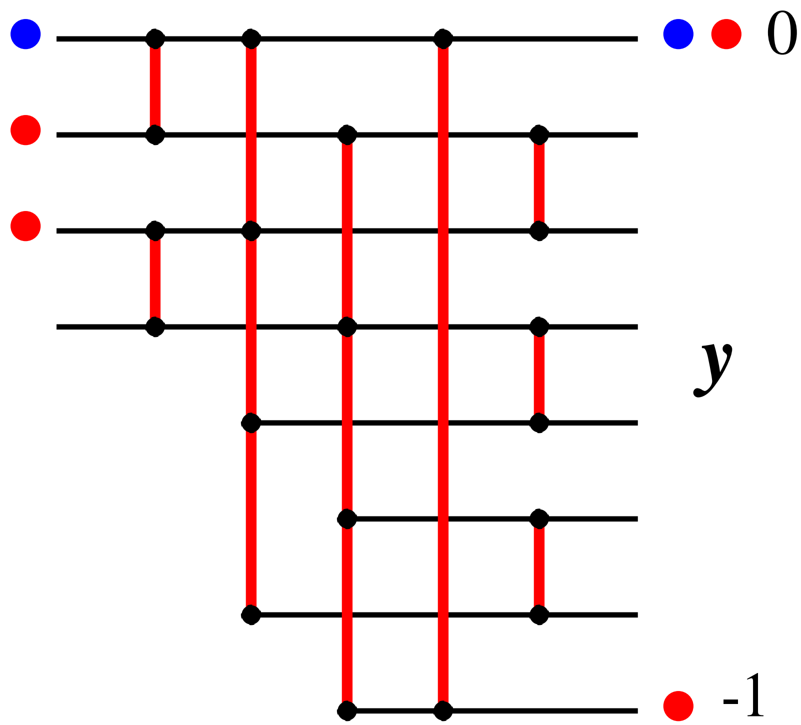
Implication:

All known counting and K-smoothing networks support decrements

Examples

These are known counting networks





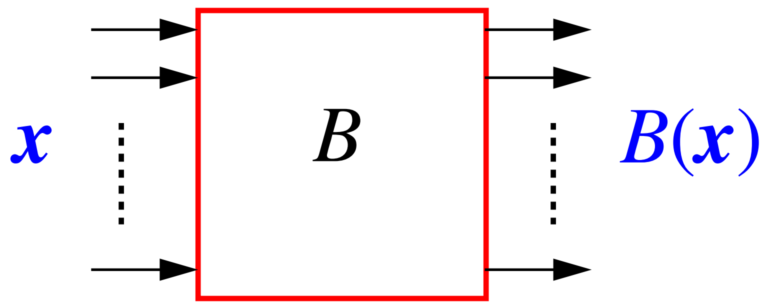
Their step property still holds:

on output vector $\boldsymbol{y}$, $0 \le y_i - y_j \le 1$ for $i < j$

The Proof

For a balancing network B with input vector $\boldsymbol{x}$:

$B(\boldsymbol{x})$ denotes the output vector



We need to show that:

if $B(\boldsymbol{x})$ has a boundedness property P
whenever $\boldsymbol{x}$ is non-negative then
$B(\boldsymbol{x})$ has the property P for any $\boldsymbol{x}$

Preliminaries: State of Balancer

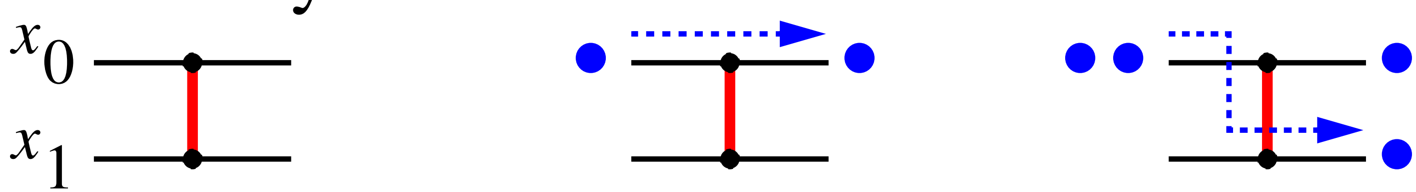
The state of balancer b with fan-out f_{out}:

- points where the next token will exit

- formally, on input vector $\boldsymbol{x}$:

$$\text{state}_b(\boldsymbol{x}) = \Sigma(\boldsymbol{x}) \bmod f_{\text{out}}$$

Initially



state = 0 state = 1 state = 0

Preliminaries: State of Balancing Network

The state of a balancing network *B*:

is the collection of the states of its balancers

A fooling pair to network *B* is:

a pair of input vectors $\boldsymbol{x}$ and $\boldsymbol{x'}$ that drive the network *B* to identical states

A null vector to network *B*:

- is any input vector $\boldsymbol{x}$ that is a fooling pair with the vector $\mathbf{0}$
- drives network *B* to its initial state

Preliminaries: Proposition 1

For any balancing network B,
any input vector $\boldsymbol{z}$,
and any fooling pair $\boldsymbol{x}$ and $\boldsymbol{x'}$:

- $\boldsymbol{x} + \boldsymbol{z}$ and $\boldsymbol{x'} + \boldsymbol{z}$ is a fooling pair

- $B(\boldsymbol{x} + \boldsymbol{z}) - B(\boldsymbol{x}) = B(\boldsymbol{x'} + \boldsymbol{z}) - B(\boldsymbol{x'})$

Proof: By induction on the depth of B

Preliminaries: Proposition 2

For any balancing network B,
any null vector $\boldsymbol{x}$,
and any integer k:

- $B(k\,\boldsymbol{x}) = k\,B(\boldsymbol{x})$
- $k\,\boldsymbol{x}$ is a null vector

<u>Proof:</u> By induction on k

Preliminaries: Proposition 3

For any balancing network *B*,
and any input vector x
such that $W(B)$ divides $\boldsymbol{x}$:

$\boldsymbol{x}$ is a null vector

$W(B)$ is the product of the fan-outs of the balancers of network *B*

Proof: By induction on the depth of *B*

Theorem 1

For any balancing network B
that has a boundedness property,
and any input vector $\boldsymbol{x}$
such that $W(B)$ divides $\boldsymbol{x}$:

$B(\boldsymbol{x})$ is a constant vector

Proof: By contradiction

Assume $B(\boldsymbol{x})$ is not constant.

There are elements a and b of $B(\boldsymbol{x})$:

$$|a - b| \geq 1$$

From the boundedness property:

$B(\boldsymbol{x})$ is a K-smoothing vector

By Proposition 3, since $W(B)$ divides $\boldsymbol{x}$:

$\boldsymbol{x}$ is a null vector

By Proposition 2, since x is a null vector:

$$B((K + 1)\,\boldsymbol{x}) = (K+1)\,B(\boldsymbol{x})$$

Subsequently, for the elements a' and b' of $B((K+1)\,\boldsymbol{x})$:

$$|a' - b'| = (K+1)|a-b| \geq K+1$$

Therefore, the vector $B((K+1)\,\boldsymbol{x})$ is at least $(K+1)$-smoothing:

a contradiction! (it is K-smoothing)

Main Theorem

For any balancing network B
that has a boundedness property P
whenever the input vector $\boldsymbol{x}$ is non-negative:

the network B has the property P for any input vector $\boldsymbol{x}$

Proof:

Given any vector x, construct vector $\boldsymbol{x'}$:

- $W(B)$ divides $\boldsymbol{x'}$
- $\boldsymbol{x'} + \boldsymbol{x} \geq \mathbf{0}$

Since $\boldsymbol{x} + \boldsymbol{x}'$ is non-negative

$B(\boldsymbol{x}' + \boldsymbol{x})$ has the property P

By Theorem 1, since $W(B)$ divides $\boldsymbol{x}'$:

$B(\boldsymbol{x}')$ is a constant vector

By proposition 3, since $W(B)$ divides $\boldsymbol{x}'$:

$\boldsymbol{x}'$ is a null vector

By proposition 1, since $\boldsymbol{x'}$ is a null vectror:

$$B(\boldsymbol{x}) = B(\boldsymbol{x'} + \boldsymbol{x}) - B(\boldsymbol{x'})$$

Subsequently, since

- $B(\boldsymbol{x'} + \boldsymbol{x})$ has the P property
- $B(\boldsymbol{x'})$ is constant,
- property P is closed under addition with a constant vector

$B(\boldsymbol{x})$ has the property P, as needed

Conclusions

We showed that:

balancing networks which satisfy a boundedness property can support both increment and decrement operations

Open problem:

Do randomized balancing networks (Aiello et al. PODC '94) support both increment and decrement operations?