Data Structures and Algorithms — CSCI 230
Chapter 1 — Review Solutions

1. Give an inductive proof showing that for all integers $n \geq 5$, $4n + 4 < n^2$.

   SOLUTION:

   Basis Case: When $n = 5$, $4n + 4 = 24$ and $n^2 = 25$. Since $24 < 25$, the basis case is proved.

   Induction Step: For $n > 5$, if $4k + 4 < k^2$ for $5 \leq k < n$ then

$$\begin{aligned}
4n + 4 &= [4(n-1) + 4] + 4 \\
&< (n-1)^2 + 4 \qquad \ldots \text{ by the Inductive Hypothesis} \\
&= n^2 - 2n + 5 \\
&< n^2 \qquad \ldots \text{since } -2n + 5 < 0 \text{ if } n > 5
\end{aligned}$$

2. Give an inductive proof showing that for all positive integers n,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$$

   SOLUTION:

   Basis Case: $n = 1$

   The left hand side is

$$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1 \cdot 3} = \frac{1}{3}.$$

   The right hand side is

$$\frac{1}{2 \cdot 1 + 1} = \frac{1}{3}.$$

   Since these are equal, the basis case is established.

   Induction Step: For $n > 2$, if $\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$ for $1 \leq k < n$ then

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{n-1} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2n-1)(2n+1)} \\
&= \frac{(n-1)}{2n-1} + \frac{1}{(2n-1)(2n+1)} \qquad \ldots \text{ by the inductive hypothesis} \\
&= \frac{1}{2n-1}\left[\frac{(n-1)(2n+1)}{2n+1} + \frac{1}{2n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2n-1}\,\frac{2n^2-n}{2n+1} \\
&= \frac{1}{2n-1}\,\frac{(2n-1)n}{2n+1} \\
&= \frac{n}{2n+1}.
\end{aligned}
$$

3. Give an inductive proof showing that for all $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

SOLUTION:
Basis Case: When $n = 1$, $\sum_{i=1}^{n} i(i!) = 1(1!) = 1$. Substituting $n = 1$ in $(n+1)! - 1$ yields $2! - 1 = 1$. Since these are equal the basis case is proved.

Induction Step: For $n \geq 2$, if $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$, for $1 \leq k < n$, then

$$
\begin{aligned}
\sum_{i=1}^{n} i(i!) &= \sum_{i=1}^{n-1} i(i!) + n(n!) \\
&= ((n-1)+1)! - 1 + n(n!) \qquad \text{by hypothesis, with } k = n-1 \\
&= n! - 1 + n(n!) \\
&= (1+n)(n!) - 1 \\
&= (n+1)! - 1
\end{aligned}
$$

4. Evaluate the following summations.

(a) $\sum_{i=0}^{100} (-1)^i$

SOLUTION:
Writing this out explicitly yields:

$$
\begin{aligned}
\sum_{i=0}^{100} (-1)^i &= (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + \ldots + (-1)^{100} \\
&= 1 + (-1) + 1 + (-1) + \ldots + 1 + (-1) + 1 \\
&= (1 + (-1) + (1 + (-1)) + \ldots + (1 + (-1)) + 1 \\
&= 1
\end{aligned}
$$

(b) $\sum_{i=5}^{2n} i$

SOLUTION:

$$\begin{aligned}
\sum_{i=5}^{2n} i &= \sum_{i=1}^{2n} i \; - \; \sum_{i=1}^{4} i \\
&= \frac{(2n)(2n+1)}{2} - 10 \qquad \text{stopping here is fine} \\
&= 2n^2 + n - 10
\end{aligned}$$

(c) $\sum_{i=0}^{n}(a^i - 2i + n)$

SOLUTION:

$$\begin{aligned}
\sum_{i=0}^{n}(a^i - 2i + n) &= \sum_{i=0}^{n} a^i - 2\sum_{i=0}^{n} i + \sum_{i=0}^{n} n \\
&= \frac{a^{n+1}-1}{a-1} - (n+1)n + (n+1)n \\
&= \frac{a^{n+1}-1}{a-1}
\end{aligned}$$

(d) $\sum_{i=0}^{n-1}\left(c + \sum_{j=0}^{i-1}(j+2)\right)$

SOLUTION:

This one's pretty involved and is a bit too much to do on a quiz...

$$\begin{aligned}
\sum_{i=0}^{n-1}\left(c + \sum_{j=0}^{i-1}(j+2)\right) &= c \cdot n + \sum_{i=0}^{n-1} \frac{i(i-1)}{2} + \sum_{i=0}^{n-1} 2i \\
&= c \cdot n + \sum_{i=0}^{n-1} \frac{i^2}{2} - \sum_{i=0}^{n-1} \frac{i}{2} + \frac{2(n-1)n}{2} \\
&= c \cdot n + \frac{(n-1)n(2n-1)}{12} - \frac{(n-1)n}{4} + (n-1)n \\
&= c \cdot n + \frac{(n-1)n}{12}\big[(2n-1) - 3 + 12\big] \\
&= c \cdot n + \frac{(n-1)n(n+4)}{12}
\end{aligned}$$

5. Prove using mathematical induction that $f_n > (3/2)^n$, for $n \geq 5$. Here, f_n is the nth Fibonacci number. Use the definition of the Fibonacci numbers given in the text. Study the example inductive proof on page 6 carefully.

SOLUTION:

Basis case: Two basis cases are needed. (This is important!) For $n = 5$, $f_5 = 8$ and $(3/2)^5 < 7.594$. For $n = 6$, $f_6 = 13$ and $(3/2)^6 < 11.391$. In each of these cases, $f_n > (3/2)^n$.

Induction hypothesis: For all $k, 5 \leq k < n$, $f_k < (3/2)^k$.

Induction step: For $n \geq 7$,

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ &> (3/2)^{n-1} + (3/2)^{n-2} && \text{Induction Hypothesis} \\ &= \frac{(3/2)^n}{3/2} + \frac{(3/2)^n}{(3/2)^2} \\ &= \frac{2}{3}(3/2)^n + \frac{4}{9}(3/2)^n \\ &= (\frac{2}{3} + \frac{4}{9})(3/2)^n \\ &= \frac{10}{9}(3/2)^n \\ &> (3/2)^n \end{aligned}$$

Hence, $f_n > (3/2)^n$.

6. Write the code necessary to accomplish the merging step in `MergeSort`. This code should follow the comments in the main `MergeSort` function.

SOLUTION:
The following is probably more terse than what you came up with on your own. Be sure you understand what it is doing. It could be made more efficient, mostly by eliminating the need for allocating `temp`.

```
T* temp = new T[high-low+1];  // scratch array for merging
int i=low, j=mid+1, loc=0;

//  while neither the left nor the right half is exhausted,
//  take the next smallest value into the temp array
while ( i<=mid && j<=high ) {
  if ( pts[i] < pts[j] ) temp[loc++] = pts[i++];
  else                   temp[loc++] = pts[j++];
}

//  copy the remaining values --- only one of these will iterate
```

```
for ( ; i<=mid; i++, loc++ ) temp[loc] = pts[i];
for ( ; j<=high; j++, loc++ ) temp[loc] = pts[j];

//  copy back from the temp array
for ( loc=0, i=low; i<=high; loc++, i++ ) pts[i]=temp[loc];
delete [] temp;
```