Data Structures and Algorithms — CSCI 230
Discrete Mathematics Overview — Part A

Introduction

During the first three weeks of the semester we will cover some techniques from discrete mathematics relevant to computer science in general and algorithm analysis in particular. This material is somewhat more extensive than offered in Chapter 1 of the Weiss text. It does not even begin to substitute for a course in discrete mathematics, however. Students, especially those interested in graduate school in computer science, should consider taking a separate discrete mathematics course during their undergraduate careers.

Sets

- A set is an **unordered** collection of distinct objects (called *elements*). These elements may be anything, including other sets.
- There are many special sets with special designations. $\boldsymbol{Z}$ is the set of integers. $\boldsymbol{N}$ is the set of natural numbers (integers beginning with 0). $\boldsymbol{R}$ or $\Re$ is the set of real numbers. The set with no elements, often called the "null set" or the "empty set," is denoted $\{\}$.
- The elements of a set may be specified in a number of ways. The simplest one to list the elements, e.g.,

  $$V = \{a, u, e, i, o\}.$$

  We use the notation $a \in V$ to say "a is an element of V". Another way to specify the elements of a set is to describe them logically using what's referred to as "set-builder notation". For example,

  $$S = \{x \mid \ x \in \boldsymbol{R} \text{ and } 0 \leq x \leq 1\}$$

  is the set of **all** real numbers in the interval from 0 to 1 inclusive.
- A set S is a *subset* of another set T if and only if each element of S is also an element of T. This is denoted by $S \subseteq T$.
- The *cardinality* of a set S, denoted by $|S|$, is the number of distinct elements of S.
- New sets may be formed from existing sets, S and T, by a variety of operations, such as union ($S \cup T$), intersection ($S \cap T$), set difference ($S - T$) and set complement $\overline{S}$. Set complement requires the notion of a "universal set" U from which set elements can be drawn.

  Most of these set operations should be familiar. The only one that might be new is set difference:

  $$S - T = \{x \mid x \in S \text{ and } x \notin T\}.$$

- Two sets, A and B, are said to be *disjoint* if $A \cap B = \{\}$.

In-class problems

Work on these problems briefly. Most should be straightforward. We will discuss the solutions in class.

1. Given $S = \{a, b\}$ and $T = \{0, a, \{1, 2\}\}$.

   Answer: There are three elements: 0, a, and $\{1, 2\}$.

   (a) List the elements of T.

   (b) List all subsets of S.
   Answer: The four subsets of S are $\{\}, \{a\}, \{b\}, \{a, b\}$.

   (c) What is $|S|$? *Answer:* 2

   (d) What is $|T|$? *Answer:* 3

   (e) What are $S \cap T$ *Answer:* {a}

   and $S - T$? *Answer:* {b}

2. Let S be the set of items on your shopping list and let T be the set of items you actually bought during a trip to the store. Describe each of the following in English.

   (a) $S \cap T$ *Answer:* The items that were on the list and I actually bought.

   (b) $T - S$ *Answer:* The items I bought that weren't on the list.

   (c) $S - T$ *Answer:* The items on the list that I failed to buy.

Functions

- A function, f, is a mapping from one set, A, called the *domain*, to another set, B, called the *co-domain*, that associates **each** element of A with a **single** element of B. We write this as

$$f : A \to B$$

  If $a \in A$ and f maps a to b, we write $f(a) = b$.

- The *range* of f is the set containing all $b \in B$ such that $f(a) = b$ for some $a \in A$.

- Special functions of interest
  - Floor: $f(x) = \lfloor x \rfloor$. This is the largest integer less than or equal to x.
  - Ceiling: $f(x) = \lceil x \rceil$. This is the smallest integer greater than or equal to x.
  - Factorial: $f(n) = n!$. Note that the domain of this function is the natural numbers.

- Exponentiation (base b): $f(x) = b^x$. Rules for manipulation:

$$\begin{aligned} b^x b^y &= b^{x+y} \\ \frac{b^x}{b^y} &= b^{x-y} \\ (b^x)^y &= b^{xy} \neq b^{(x^y)} \end{aligned}$$

- Logarithm (base $b > 0$): $f(x) = \log_b x$. This means, $y = \log_b x$ if $b^y = x$. If b is unspecificed, 2 is assumed. Rules for manipulation:

$$\begin{aligned} \log_a b &= \tfrac{\log_c b}{\log_c a} && \text{for } c > 0 \quad \text{(Change of Base Law)} \\ \log xy &= \log x + \log y \\ \log(x^y) &= y \log x \neq (\log x)^y \\ \log x &< x && \text{for all } x > 0 \end{aligned}$$

In-Class Problems on Functions

1. What are
   (a) $5!$ *Answer:* 120

   (b) $\lceil 2 \rceil$, $\lceil 2.1 \rceil$, $\lceil -2.1 \rceil$,
   Answer: $2, 3, -2$

   (c) $\lfloor 2 \rfloor$, $\lfloor 2.1 \rfloor$, $\lfloor -2.1 \rfloor$?
   Answer: $2, 2, -3$

2. What is the derivative of $\log_b x$?
   Hint: Recall that the derivative of the natural logarithm is $1/x$.

   Use the Change of Base Law given above:

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad \text{for } c > 0$$

Answer:

$$\frac{d}{dx}(\log_b\ x) = \frac{d}{dx}(\frac{\log_e\ x}{\log_e\ b}) = \frac{1}{\log_e\ b}\ \frac{d}{dx}(\log_e\ x)$$
$$= \frac{1}{x\ \log_e\ b}$$

Summations

- A summation is a short-hand notation to describe the addition of the terms of a sequence:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \ldots + f(n)$$

  The variable i is called the index variable. Most often $m = 0$ or $m = 1$. Sometimes a summation is easiest to think about by thinking about the `for` loop you would write to calculate it.

- Special summation formulas:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$
$$\sum_{i=0}^{n} a^i = \tfrac{a^{n+1}-1}{a-1} \qquad \text{for } a \neq 1$$
$$\sum_{i=1}^{n} i = \tfrac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \tfrac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} \frac{1}{i} \approx \log_e n$$

- Simple rules for manipulating summations. These rules may perhaps be best understood by writing the summation out long hand and then using standard rules of algebra.

$$\sum_{i=m}^{n}[f(i)+g(i)] = \left(\sum_{i=m}^{n} f(i)\right) + \left(\sum_{i=m}^{n} g(i)\right)$$

$$\sum_{i=m}^{n} a \cdot f(i) = a \cdot \sum_{i=m}^{n} f(i)$$

$$\sum_{i=m}^{n} a = a \cdot (n-m+1)$$

$$\sum_{i=m}^{n} f(i) = \sum_{i=0}^{n} f(i) - \sum_{i=0}^{m-1} f(i)$$

$$\sum_{i=m}^{n} f(i) = \sum_{j=0}^{n-m} f(j+m)$$

In these formulas, a stands for an arbitrary but fixed constant (i.e. something that doesn't change when i changes).

Problems on summations

1. Write each of the following as a summation.

   (a) $5+10+15+20+\ldots+1000$
   Answer: $\sum_{i=1}^{200} 5i$

   (b) $3+8+13+18+\ldots+98$
   Answer: $\sum_{i=0}^{19}(3+5i)$

   (c) $4+8+16+32+\ldots+1024$
   Answer: $\sum_{i=2}^{10} 2^i$

2. Derive a formula for each of the following that does not involve a summation and does not involve i. Use the rules for manipulation and the special formulas.

   (a) $\displaystyle\sum_{i=1}^{n} 4i.$
   Answer: $4\sum_{i=1}^{n} i = 2n(n+1)$.

   (b) $\displaystyle\sum_{i=3}^{n}(5i+4).$

Answer:

$$\begin{aligned} 5\sum_{i=3}^{n} i + 4\sum_{i=3}^{n} 1 &= \frac{5n}{2}(n+1) - (5+10) + 4(n-2) \\ &= \frac{1}{2}(5n^2 + 13n - 46). \end{aligned}$$

(c) $\displaystyle\sum_{j=0}^{k}(4^j + k)$.

Answer:

$$\sum_{j=0}^{k} 4^j + k\sum_{j=0}^{k} 1 = \frac{4^{k+1}-1}{3} + k(k+1)$$

(d) $\displaystyle\sum_{i=0}^{N}(2 + \sum_{j=0}^{i} j)$

Answer:

$$\begin{aligned} 2\sum_{j=0}^{N} 1 + \sum_{i=0}^{N} \frac{i(i+1)}{2} &= 2(N+1) + \frac{1}{2}\sum_{i=0}^{N} i^2 + \frac{1}{2}\sum_{i=0}^{N} i \\ &= 2(N+1) + \frac{1}{12}N(N+1)(2N+1) + \frac{1}{4}N(N+1) \\ &= 2N + 2 + \frac{1}{12}(2N^3 + 3N^2 + N) + \frac{1}{4}N^2 + \frac{1}{4}N \\ &= \frac{1}{6}N^3 + \frac{1}{2}N^2 + \frac{7}{3}N + 2 \end{aligned}$$

3. (Extra challenge!) Derive a formula for the following that does not involve a summation:

$$\sum_{i=1}^{N} \frac{i}{2^i}.$$

Hint: use the technique given in class for proving that

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}.$$

Proofs

- A proof is a clear, logical demonstration of the truth of an assertion based on previously known facts (established propositions, lemmas and theorems).

- In writing a proof, the first step is to carefully determine what the assertion says, including the meanings of all terms used.

  For example, you cannot prove that 2 is the only even prime number without understanding what it means for a number to be "even" and to be "prime".

  Once you understand what is to be proved, the next step is to write the actual proof. This is a bit of an art form, which we will not have time to investigate thoroughly. But, ...

- Proof techniques we will explore:
  - Direct proof. Starting from the assumptions — e.g. n is an even positive integer — construct a series of deductions leading to the claim — e.g. n is not prime.
  - Proof by example or counter-example. This is acceptable when proving that there is a member of a set satisfying an assertion, or when proving that not all members of a set satisfy an assertion. But it is not possible to use this proof technique to prove that all members of an infinite set satisfy an assertion.
  - Proof by contradiction. This is done by assuming that the assertion to be proved is false and showing that the assumption leads to a statement that must be both true and false (a contradiction).
  - Proof by induction. This technique is closely related to data structure and algorithm analysis and is covered in detail below.

In-Class Exercises on Proofs

1. Is it true that $n^3 > 2^n$ for all integers $n > 1$? Prove your result.

   Answer: The assertion is false, because for $n = 10$, $n^3 = 1000$ and $2^n = 1024$. We say that $n = 10$ is a counterexample to the assertion. (Any value of $n \geq 10$ can serve as a counterexample.)

2. Prove that 2 is the only even prime number.

   Answer: Proof: By contradiction. Let p be a prime > 2 and suppose p is even. Then $p = 2k$ for some $k > 1$, hence p is the product of two factors neither of which is 1, contradicting the assumption it was prime.

Continue to *Part B*