Models of Computation, HW3 Solutions

Standard mathematical notation: $\therefore$ implies that, $\exists$ there exists, $\forall$ for each, $\ni$ such that.

Problem 1: Prove the following are not regular

A (122/4b): $L = \{a^n b^l a^k : k \neq n+l \}$

Consider $L' = \{a^n b^l a^k : k = n+l \}$
By construction, $L \cup L' = \{a^*b^*a^*\}$ and $L \cap L' = \varnothing$
$\therefore L' = \{w : w \notin L, w \in \{a^*b^*a^*\}\} = L^c \cap \{a^*b^*a^*\}$ (L^c == L complement)

The language $\{a^*b^*a^*\}$ is regular (since $a^*b^*a^*$ is a regular expression).

Furthermore, since regular languages are closed under complement and intersection, we have:
If L is regular **then** L^c is regular **then** $L^c \cap \{a^*b^*a^*\} = L'$ is regular.

Therefore, if L' is not regular **then** L is not regular either. It suffices to show that L' is not regular.

Assume (for contradiction) that L' is regular.

Since L' is regular and infinite we have from the pumping lemma that there exists m such that:
$\forall\ w \in L'$ with $|w| \geq m$, $\exists$ x, y, z such that:
$|xy| \leq m$, $|y| = k > 0$, $w = xyz$ and $xy^i z \in L'$, $\forall\ i \geq 0$.

Let $w = xyz = a^m b a^{(m+1)} \in L'$. By construction, $y = a^k$ for some $k \in [1, m]$.
Consider $xy^2z \in L'$
$a^{(m+k)} b a^{(m+1)} \in L'$
$\therefore m+k+1 = m+1$ (by definition of L')
$\therefore k = 0$

but $k > 0$ (from pumping lemma)
which is a contradiction
$\therefore L'$ is not regular.
$\therefore$ L is not regular.

B (122/4g): $L = \{wwww^R : w \in \{a,b\}^*\}$

Note: $\forall$ w, $|w| = |w^R|$
$\therefore \forall\ u\ (= wwww^R) \in L, \exists\ c \geq 0$ such that $|u| = 4|w| = 4c$

Assume (for contradiction) that L is regular.

Since L is regular and infinite we have from the pumping lemma that there exists m such that:
$\forall\ u \in L$ with $|u| \geq m$, $\exists$ x, y, z such that
$|xy| \leq m$, $|y| = k > 0$, $u = xyz$ and $xy^i z \in L\ \forall\ i \geq 0$.

Let $w = a^m b$ ($u = xyz = a^m b a^m b a^m b b a^m$). By construction, $y = a^k$ for some $k \in (0,m)$.
Since $|xy| \leq m$ and xy is a substring a^m
$\therefore y = a^k$
Consider $u' = xy^2z \in L$
$u' = a^{(m+k)} b a^m b a^m b b a^m \in L$
$\therefore$there exists string w'such that $u' = w'w'w'(w')^R$
We can assume $k = 4n$, since if $k \neq 4n$ for some n, then $|u'| \neq 4c$ and $\therefore u' \notin L$

We can further assume $n > 0$, since if $n = 0$, $k = 0$, and $\therefore |y| = 0$, which is a contradiction
Since $|u| = 4m+4$, $|u'| = 4m+4+k$, $|w'| = m+1+n$
$\forall\ n \geq 1,\ 3n > 1$
$\therefore\ m + 1 + n < m + 3n + n = m + 4n = m + k$
$\therefore\ w' = a^{(m+n+1)}$ (first m+n+1 characters of u′) $\subset$ (first m+k characters of u′) $= a^{(m+k)}$
$\therefore\ u' = w'w'w'(w')^R = a^{4(m+n+1)}$
which is a contradiction, since u′ is known to contain 4 'b's
$\therefore$ L is not regular.

Problem 2: Are the following regular?
A (122/5b): $L = \{a^n : n \text{ is not prime}\}$

No.

Consider $L^c = \{a^n : n \text{ is prime}\}$. ($L^c$ is the coplement of L).

From the closure of regular languages under complement we have:
If L is regular **then** L^c is regular.

Therefore, if L^c is not regular **then** L is not regular either. It suffices to show that L^c is not regular.

Assume (for contradiction) that L^c is regular.

Since L^c is regular and infinite we have from the pumping lemma that the exists m such that:
$\forall\ w \in L^c$ with $|w| \geq m$, $\exists$ x, y, z such that:
$|xy| \leq m$, $|y| = k > 0$, $w = xyz$ and $xy^iz \in L^c\ \forall\ i \geq 0$.

Let $p > m$ be prime ; then $w = xyz = a^p \in L^c$
By observation, y must be a^k
Consider $xy^{(p+1)}z \in L^c$
$a^{(p+kp)} \in L^c$
$\therefore$ (p+kp) is prime (by definition of L^c)

but (p+kp) = p*(k+1)
$\therefore$ (p+kp) is not prime
which is a contradiction
$\therefore$ L^c is not regular

B (122/5d): $L = \{a^n : n = 2^k \text{ for some } k\}$

No.

Assume (for contradiction) that L is regular.

Since L is regular and infinite we have from the pumping lemma that there exists m such that:
$\forall\ w \in L$ with $|w| \geq m$, $\exists$ x, y, z such that:
$|xy| \leq m$, $|y| = k > 0$, $w = xyz$ and $xy^iz \in L\ \forall\ i \geq 0$.

Let $c = 2^j > m$; then $w = xyz = a^c \in L$
By observation, y must be a^k
Consider $xy^2z \in L$
$a^{(c+k)} \in L$
$\therefore\ \exists\ q\ \ni\ 2^q = c+k$ (by definition of L)

but $k \leq m < c$,
$\therefore\ 2^j = c < c+k < 2c = 2*2^j = 2^{(j+1)}$

$\therefore$ $!\exists\ q \ni 2^q = c+k$
which is a contradiction
$\therefore$ L is not regular

Problem 3: Which of the following are regular?
A (123/9c): $L = \{a^n b^l : n \text{ is an integral multiple of } l\}$

No.

We can use the pumping lemma.

Assume (for contradiction) that L is regular. By definition,
$\exists\ m \ni \forall\ w \in L, |w| \geq m, \exists\ x, y, z \ni |xy| \leq m, |y| = k > 0, w = xyz$ and $xy^i z \in L\ \forall\ i \geq 0.$
Let $w = xyz = a^m b^m \in L$ $(m = 1*m)$
Since $|xy| \leq m$, $xy \subset a^m$
$\therefore$ $y = a^k$
Consider $xy^2z \in L$
$a^{(m+k)} b^m \in L$
$\therefore$ $\exists\ n \ni (m+k) = n*m$ (by definition of L)

but $k < m$
$\therefore$ $1m = m < m+k < m+m = 2m$
$\therefore$ $!\exists\ n \ni (m+k) = n*m$
which is a contradiction
$\therefore$ L is not regular

B: $L = \{a^n b^l : (n-l) \text{ is a multiple of } 3\}$
Yes.

The following equivalent statements are used to simplify the problem:
(n-l) is a multiple of 3
$\exists\ x \ni n-l = 3x + 0$
$n-l \equiv 0 \pmod 3$
$n \equiv l \pmod 3$
The last of these has 3 cases:
$n \equiv l \equiv 0 \pmod 3$ $n \equiv l \equiv 1 \pmod 3$ $n \equiv l \equiv 2 \pmod 3$
Note that the extended regular expression $\{ (c^x)^* c^y : x > y \}$ represents $\{ c^n : n \equiv y \pmod x \}$
(this is only an *extended* regular expression when x and y are left as exponents of c; when expanded out to { (cccc…c)*ccc…c } , it is a normal regular expression)
Since $a^n b^l$ is simply a concatenation of two such expressions, the union of the 3 cases is:
$L = L(\ ((aaa)^*\ (bbb)^*) + ((aaa)^*a\ (bbb)^*b) + ((aaa)^*aa\ (bbb)^*bb)\ )$
$\therefore$ L is regular
note: the above can also be simplified to: $(aaa)^*(\ \lambda + ab + aabb)(bbb)^*$

C (123/9g): $L = \{a^n b^l : |n-l| = 2\}$

No.

We use the pumping lemma.

Assume (for contradiction) that L is regular. By definition,
$\exists\ m \ni \forall\ w \in L, |w| \geq m, \exists\ x, y, z \ni |xy| \leq m, |y| = k > 0, w = xyz$ and $xy^i z \in L\ \forall\ i \geq 0.$
Let $w = xyz = a^m b^{(m-2)} \in L$ $(\ |(m) - (m-2)| == 2)$
Since $|xy| < m$, $xy \subset a^m$
$\therefore$ $y = a^k$

Consider $xy^2z \in L$

$a^{(m+k)}b^{(m-2)} \in L$

$\therefore |(m+k) - (m-2)| = |k + 2| = k + 2 = 2$ (by definition of L)

$\therefore k = 0$

but $k > 0$ (by definition of regular, above)

which is a contradiction

$\therefore$ L is not regular

D: $L = \{a^nb^l : n + l = 100\}$

Yes.

L is finite (101 elements), and all finite languages are regular; therefore L is regular.

Problem 4: Find a Context-Free Grammar for the following:

A (134/8c): $L = \{a^nb^mc^k : k = n + m\}$

Let G be the grammar with productions:

$S \rightarrow aSc \mid B$

$B \rightarrow bBc \mid \lambda$

Claim: L(G) = L

Proof:

Consider the following derivation:

$S \rightarrow^* a^nSc^n \rightarrow a^nBc^n \rightarrow^* a^nb^mBc^mc^n \rightarrow a^nb^mc^{(n+m)}$

(where the first $\rightarrow^*$ applies $S \rightarrow aSc$ n times, the second $B \rightarrow bBc$ m times)

Since all words in L(G) must follow this pattern in their derivations, it is clear that $L(G) \subseteq L$

Consider $w \in L$, $w = a^nb^mc^{(n+m)}$ for some $n, m \geq 0$

The derivation $S \rightarrow^* a^nSc^n \rightarrow a^nBc^n \rightarrow^* a^nb^mBc^mc^n \rightarrow a^nb^mc^{(n+m)}$ clearly produces w for any n, m.

$\therefore L \subseteq L(G)$

$\therefore L = L(G)$

$\therefore$ G is a CFG for L

B (134/8c): $L = \{a^nb^mc^k : k \neq n + m\}$

Let G be the grammar with productions:

$S \rightarrow EcC \mid aAE \mid AU$

$A \rightarrow aA \mid \lambda$

$B \rightarrow bB \mid \lambda$

$C \rightarrow cC \mid \lambda$

$E \rightarrow aEc \mid F$

$F \rightarrow bFc \mid \lambda$

$U \rightarrow aUc \mid V$

$V \rightarrow bVc \mid bB$

Note: $L(E) = \{a^nb^mc^k : k = n + m\}$ (from 4(a))

Let $L1 = \{a^nb^mc^k : k < n + m\}$, $L2 = \{a^nb^mc^k : k > n + m\}$

Claim: $L(G) = L = L1 \cup L2$

Proof:

Consider the leftmost productions following $S \rightarrow EcC$

$S \rightarrow EcC \rightarrow^* a^nb^mc^{(n+m)}cC \rightarrow^* a^nb^mc^{(n+m)}cc^kC \rightarrow a^nb^mc^{(n+m)}cc^k = a^nb^mc^{(n+m+k+1)}$

(where the first $\rightarrow^*$ follows from 4(a), the second from k repetitions of $\{ C \rightarrow cC \}$)

Since $(n+m) < (n+m+k+1)$, $L(EcC) \subseteq L2 \subset L$

Consider the leftmost productions following $S \rightarrow aAE$

$S \rightarrow aAE \rightarrow^* aa^kAE \rightarrow aa^kE \rightarrow^* aa^ka^nb^mc^{(n+m)}c = a^{(n+k+1)}b^mc^{(n+m)}$

(where the first $\rightarrow^*$ follows from k repetitions of $\{ A \rightarrow aA \}$, the second from 4(a))

Since (n+m+k+1) > (n+m), $L(aAE) \subseteq L1 \subset L$

Consider the leftmost productions following $S \rightarrow AU$

$S \rightarrow AU \rightarrow^* a^kAU \rightarrow a^kU \rightarrow^* a^ka^nUc^n \rightarrow a^ka^nVc^n \rightarrow^* a^ka^nb^mVc^mc^n \rightarrow a^ka^nb^mbBc^mc^n$
$\rightarrow^* a^ka^nb^mbb^jBc^mc^n \rightarrow a^ka^nb^mbb^jc^mc^n = a^{(k+n)}b^{(m+j+1)}c^{(m+n)}$
(where the first, second, third and fourth $\rightarrow^*$ are k, n, m, and j repetitions of
{ $A \rightarrow aA$ }, { $U \rightarrow aUc$ }, { $V \rightarrow bVc$ }, and { $B \rightarrow bB$ }, respectively)
Since $((k+n)+(m+j+1)) > (m+n)$, $L(AU) \subseteq L1 \subset L$

$\therefore L(G) = L(S) = L(EcC) \cup L(aAE) \cup L(AU) \subseteq L1 \cup L2 = L$

Consider $w \in L1$, $w = a^nb^mc^{(n+m+k)}$ for some n, m, k, with $k > 0$
Then w has the leftmost derivation:

$S \rightarrow EcC \rightarrow^* a^nb^mc^{(n+m)}cC \rightarrow^* a^nb^mc^{(n+m)}cc^{(k-1)}C \rightarrow a^nb^mc^{(n+m)}cc^{k(k-1)} = a^nb^mc^{(n+m+k)} = w$

$\therefore L2 \subseteq L(G)$

Consider $w \in L2$, $w = a^{(n+j)}b^{(m+k)}c^{(n+m)}$ for some j, k, m, n, with $j+k > 0$
If $k = 0$, w has the leftmost derivation:

$S \rightarrow aAE \rightarrow^* aa^{(j-1)}AE \rightarrow aa^{(j-1)}E \rightarrow^* aa^{(j-1)}a^nEc^n \rightarrow aa^{(j-1)}a^nFc^n \rightarrow^* aa^{(j-1)}a^nb^mFc^mc^n$
$\rightarrow aa^{(j-1)}a^nb^mc^mc^n = a^{(j+n)}b^{(0+m)}c^{(m+n)} = w$

$\therefore L1 (k = 0) \subseteq L(G)$

If $k > 0$, then w has the leftmost derivation:

$S \rightarrow AU \rightarrow^* a^jAU \rightarrow a^jU \rightarrow^* a^ja^nUc^n \rightarrow a^ja^nVc^n \rightarrow^* a^ja^nb^mVc^mc^n \rightarrow a^ja^nb^mbBc^mc^n$
$\rightarrow a^ja^nb^mbb^{(k-1)}Bc^mc^n \rightarrow a^ja^nb^mbb^{(k-1)}c^mc^n = a^{(j+n)}b^{(m+k)}c^{(m+n)} = w$

$\therefore L1 (k > 0) \subseteq L(G)$

$\therefore L1 \subseteq L(G)$
$\therefore L = L1 \cup L2 \subseteq L(G)$
$\therefore L = L(G)$

Problem 5 (170/3): Convert the following to Chomsky Normal Form:
$S \rightarrow aSaA \mid A$
$A \rightarrow abA \mid b$

$S \rightarrow I_1I_2 \mid TA \mid b$
$A \rightarrow TA \mid b$
$I_1 \rightarrow T_1S$
$I_2 \rightarrow T_1A$
$T \rightarrow T_1T_2$
$T_1 \rightarrow a$
$T_2 \rightarrow b$

Method:

((1) : skipping : G has no λ-productions)
(2) : Create productions for each terminal and substitute
$T_1 \rightarrow a$, $T_2 \rightarrow b$
(3) : Split productions of length > 2, creating intermediary productions of length 2
$S \rightarrow T_1ST_1A$ ➔ { $S \rightarrow I_1I_2$, $I_1 \rightarrow T_1S$, $I_2 \rightarrow T_1A$ }
$A \rightarrow T_1T_2A$ ➔ { $A \rightarrow TA$, $T \rightarrow T_1T_2$ }
(4) : remove unit productions
$S \rightarrow A$ ➔ { $S \rightarrow TA \mid b$ }