Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D $w \in D$

Result Region: S $f(w) \in S$

A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

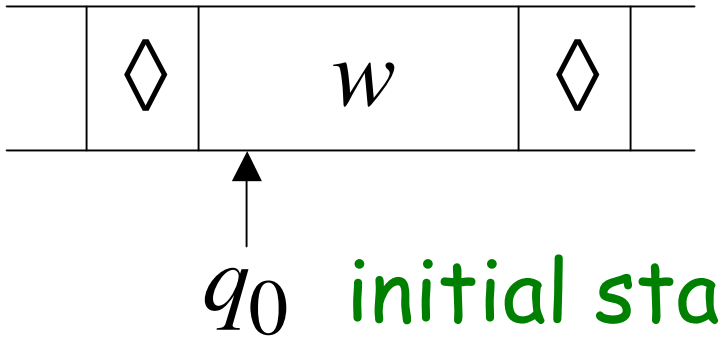
We prefer **unary** representation:

easier to manipulate

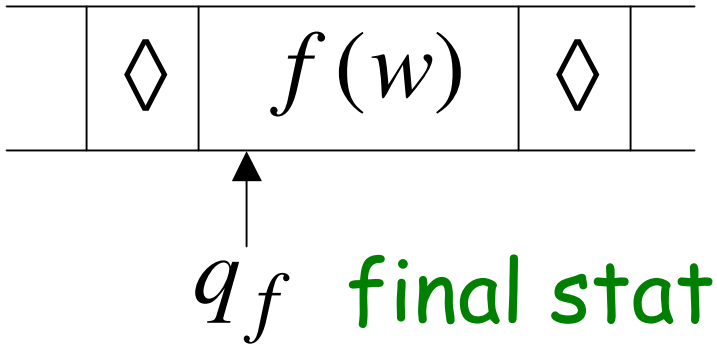
Definition:

A function f is computable if there is a Turing Machine M such that:

Initial configuration



Final configuration



For all $w \in D$ Domain

In other words:

A function f is computable if there is a Turing Machine M such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Example

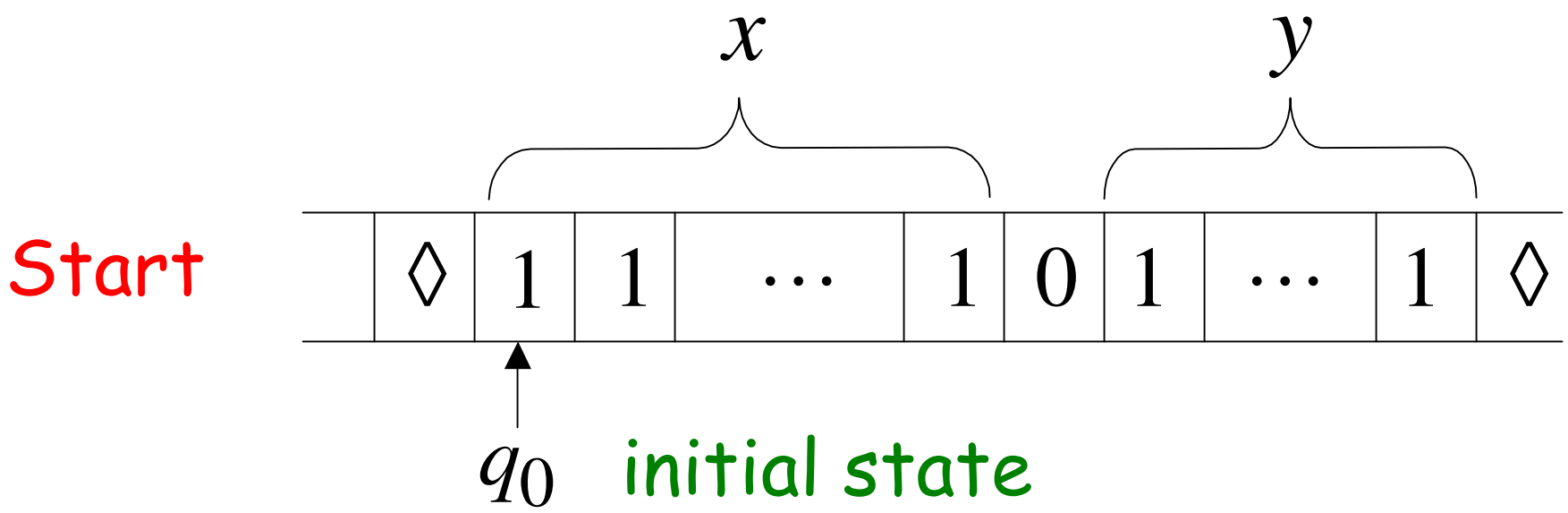
The function $f(x, y) = x + y$ is computable

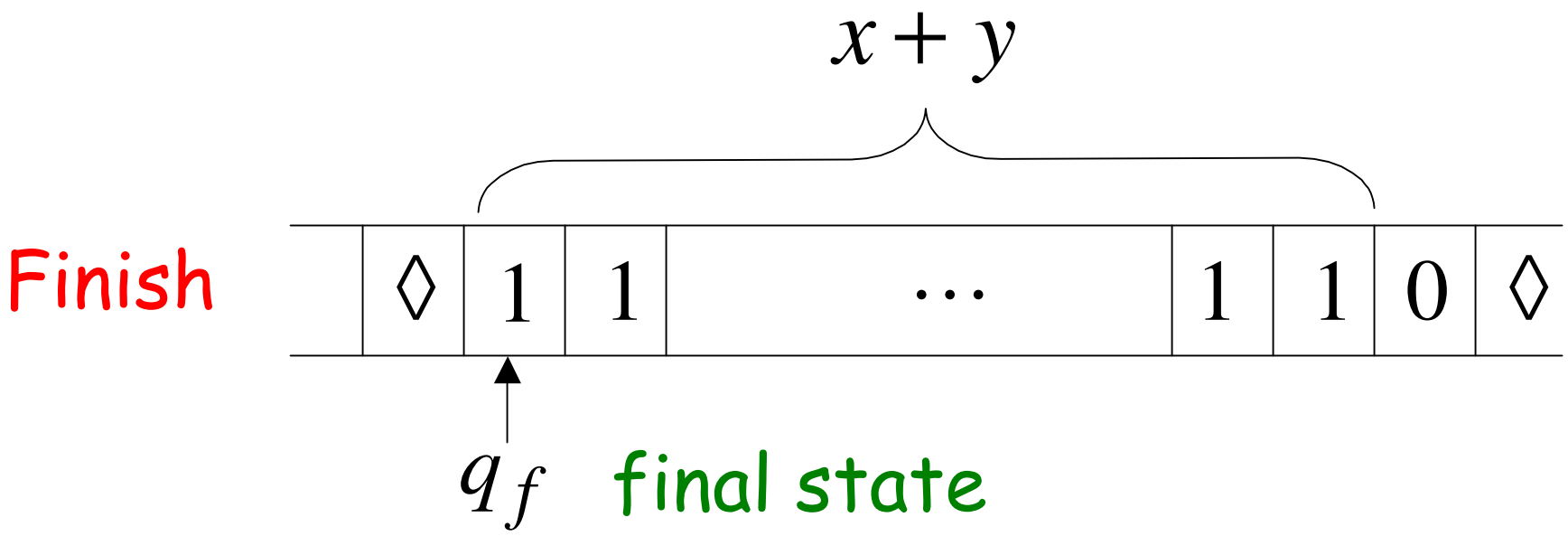
x, y are integers

Turing Machine:

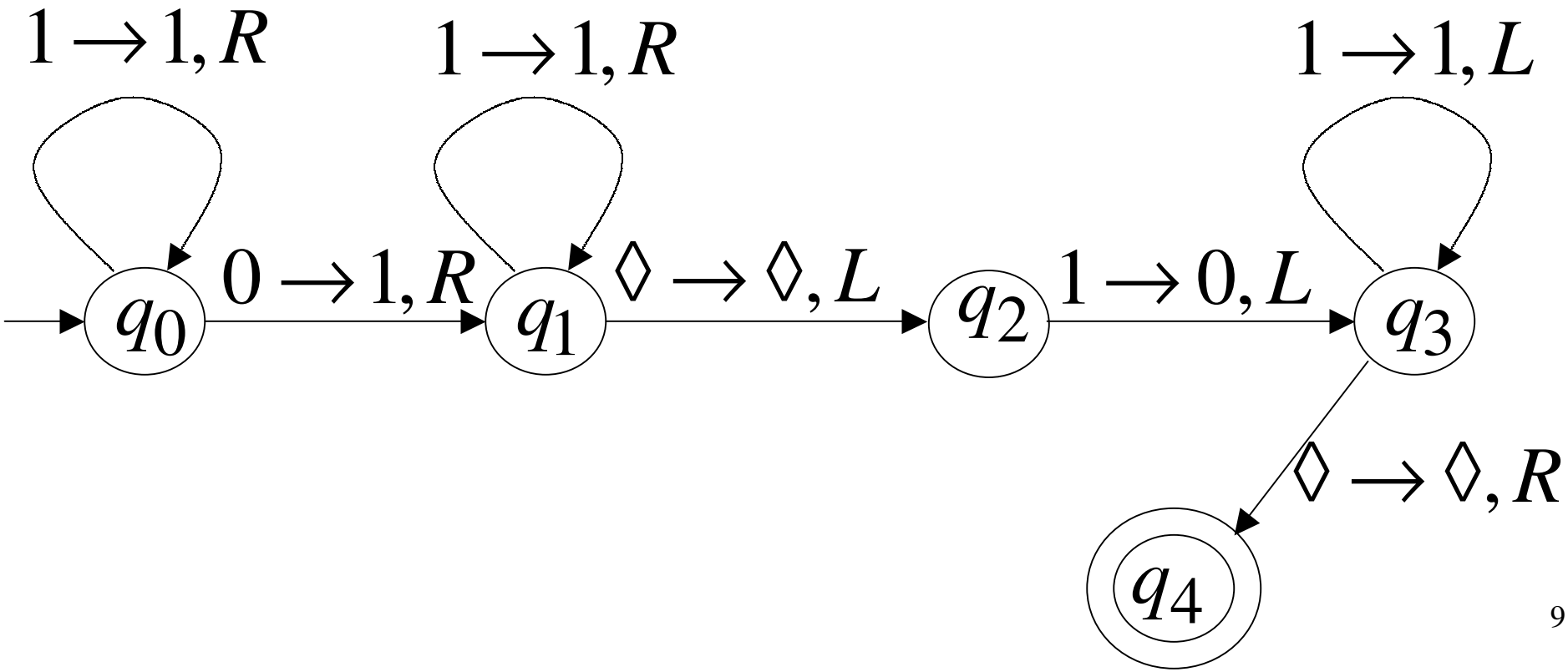
Input string: $x0y$ unary

Output string: $xy0$ unary


Start
x
y
◊ 1 1 ⋯ 1 0 1 ⋯ 1 ◊
q_0 initial state


Finish
x + y
◊ 1 1 ⋯ 1 1 0 ◊
q_f final state

Turing machine for function $f(x, y) = x + y$

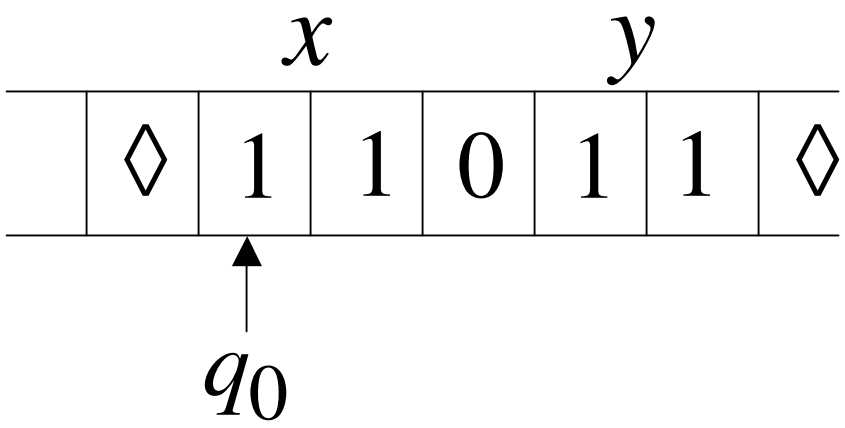


Execution Example:

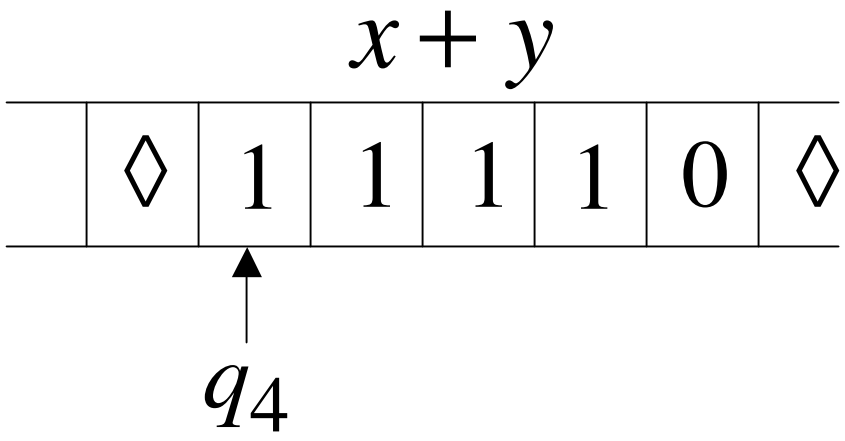
$x = 11$ (2)

$y = 11$ (2)

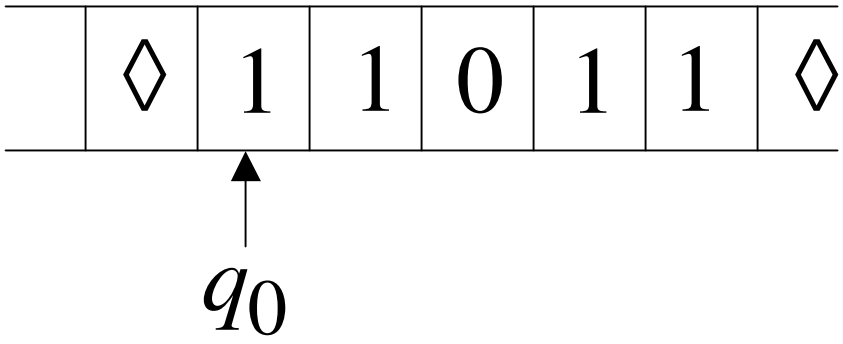
Time 0

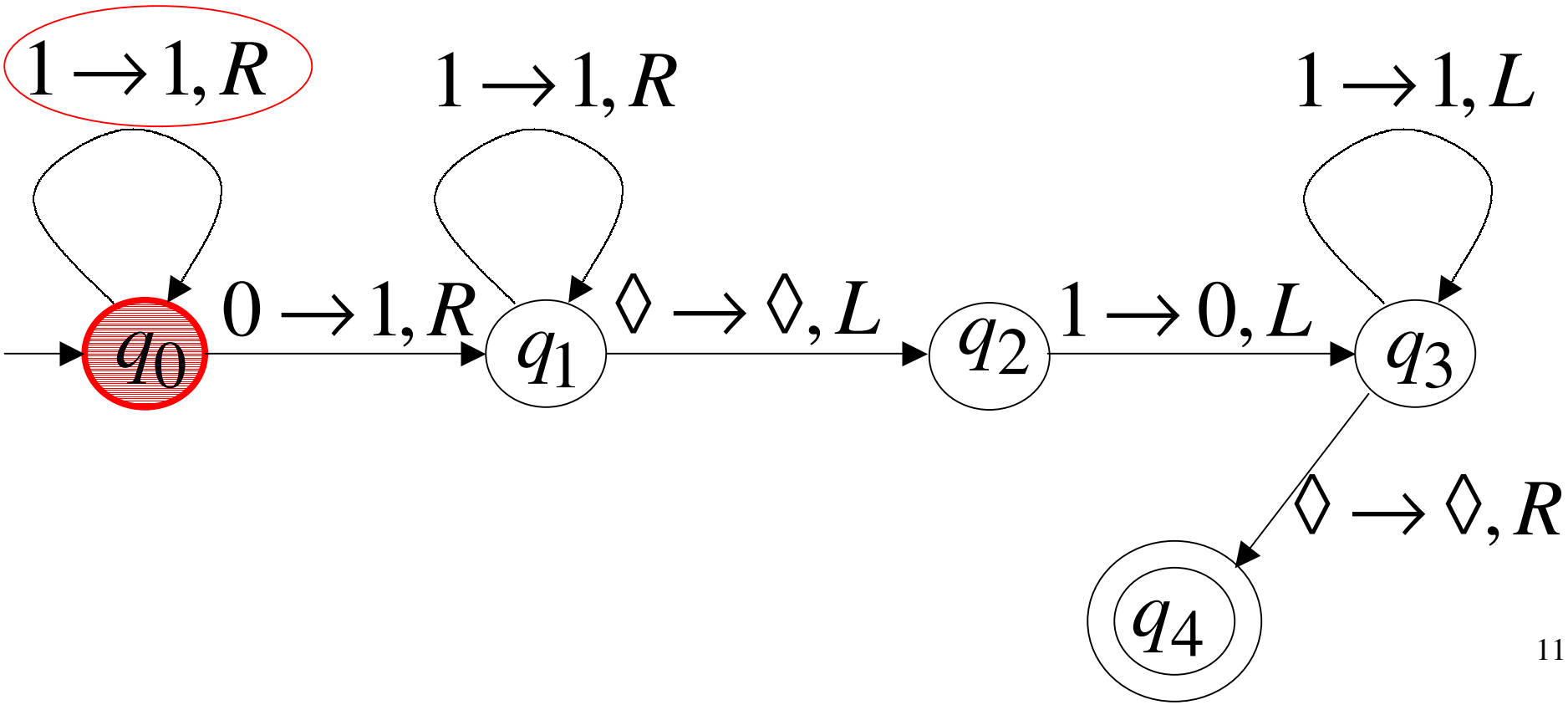


Final Result



Time 0


◊ 1 1 0 1 1 ◊
q_0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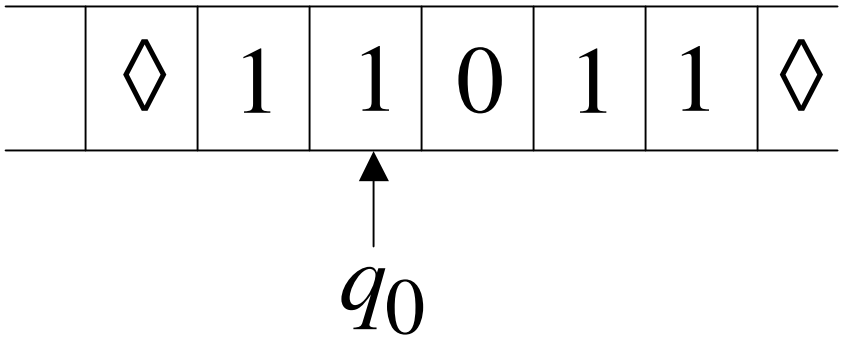
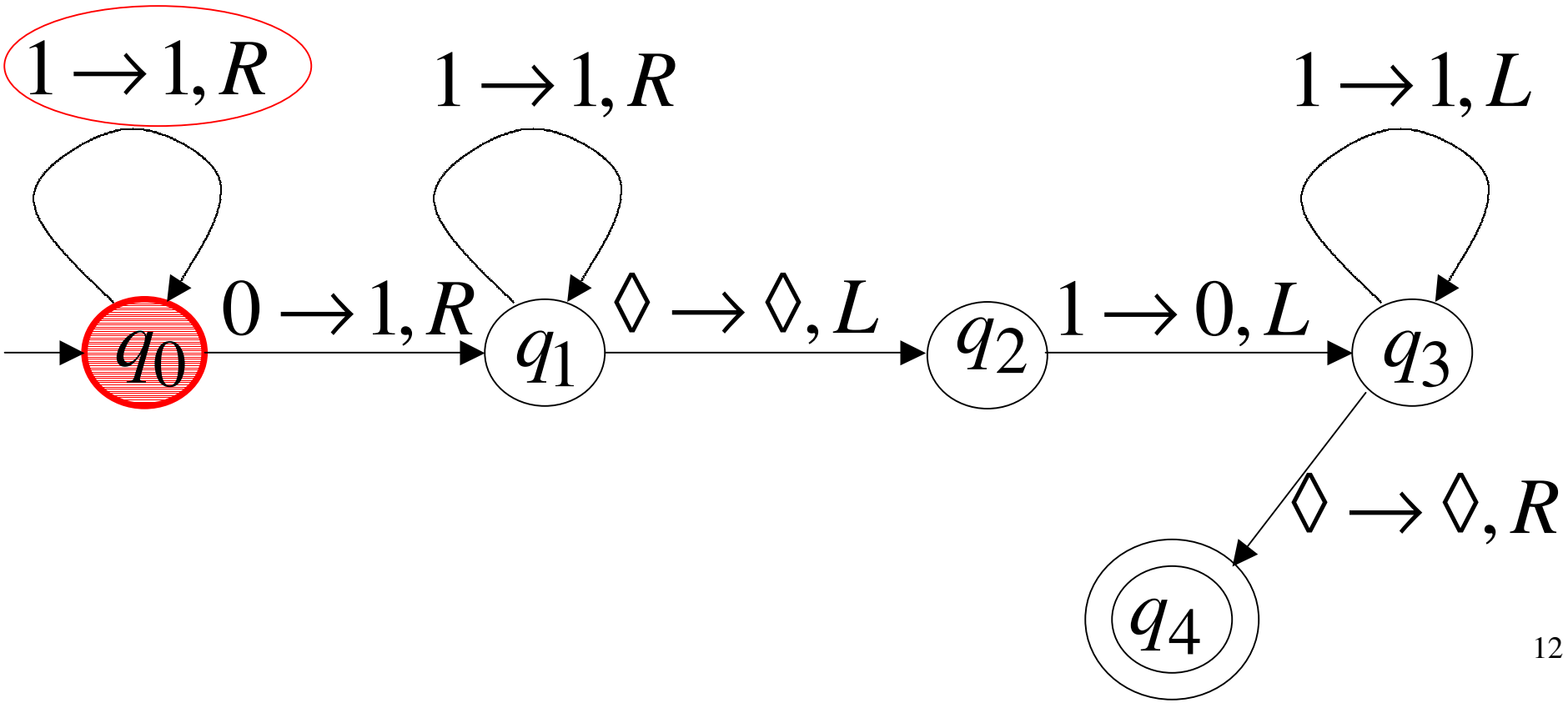
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 1


◊ 1 1 0 1 1 ◊
q_0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$1 \to 1, R$
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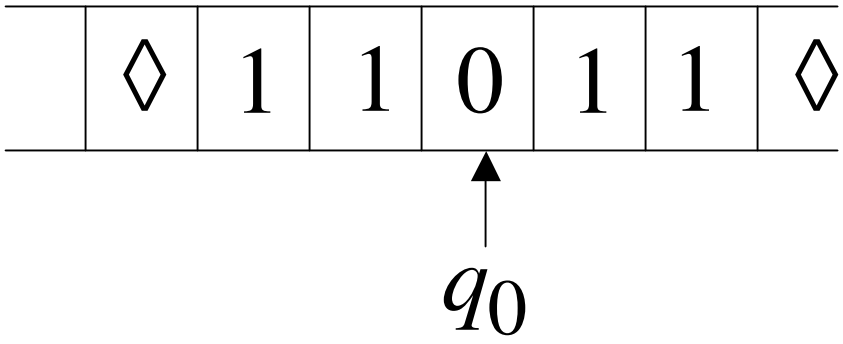
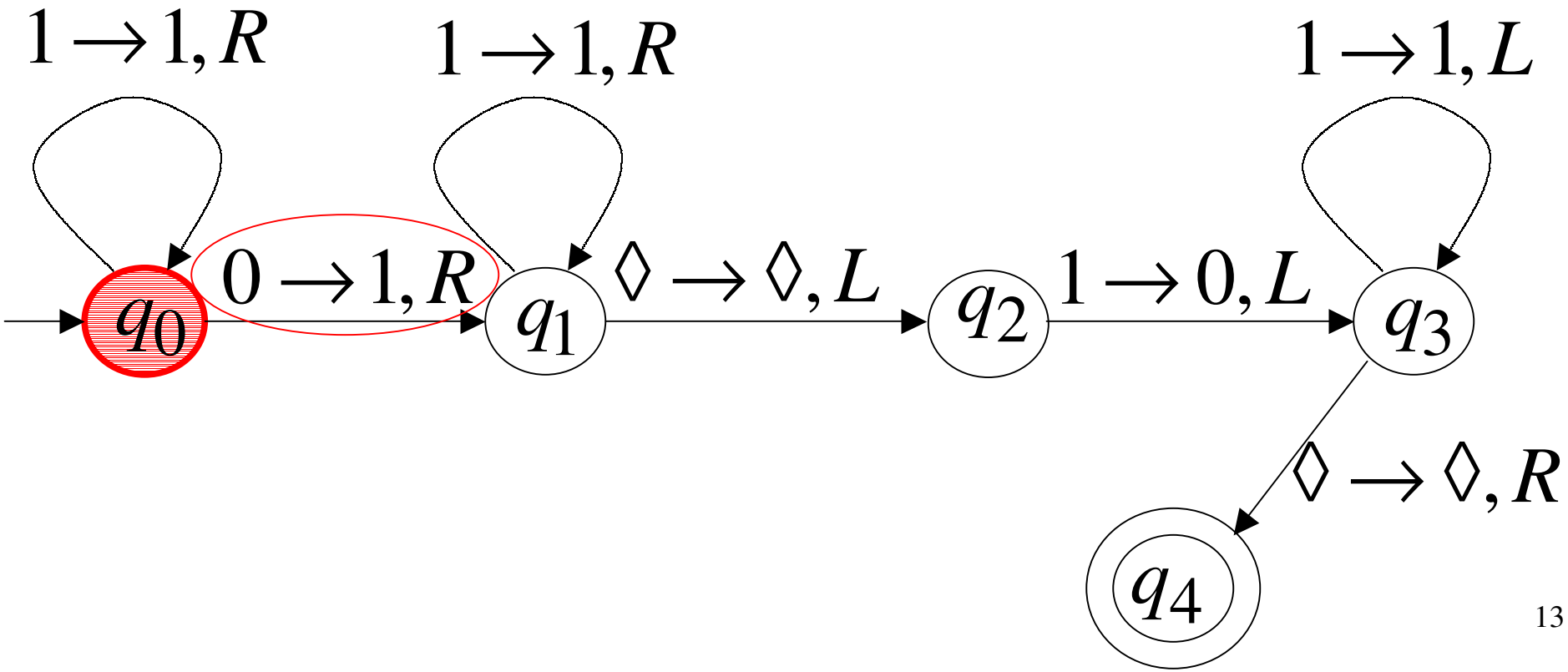
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 2


◊ 1 1 0 1 1 ◊
q_0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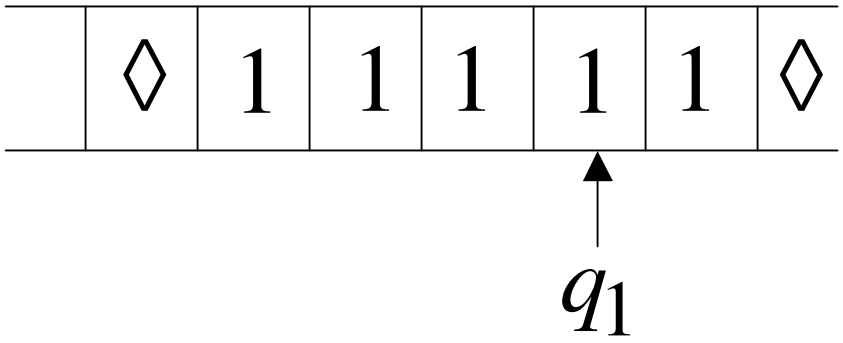
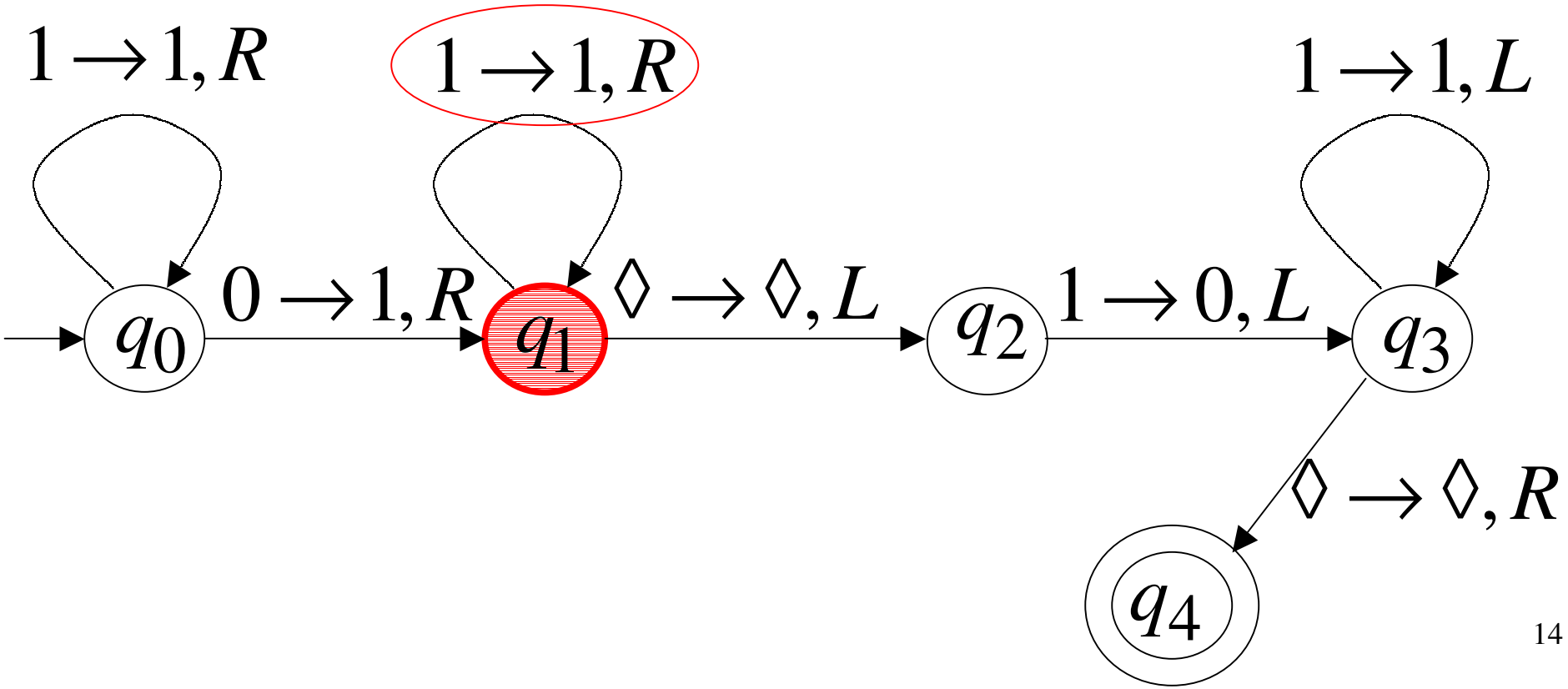
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 3


◊ 1 1 1 1 1 ◊
q_1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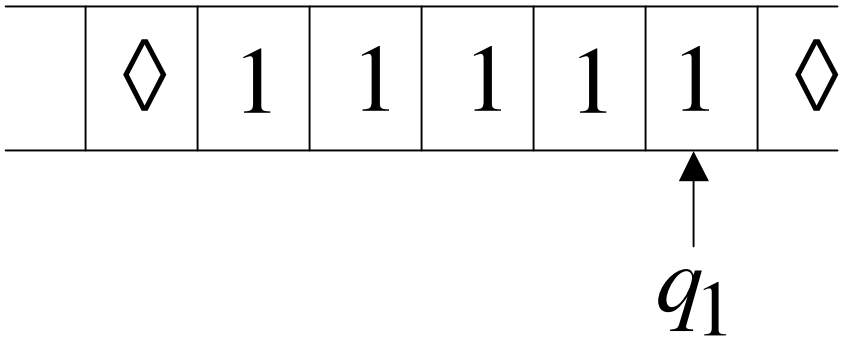
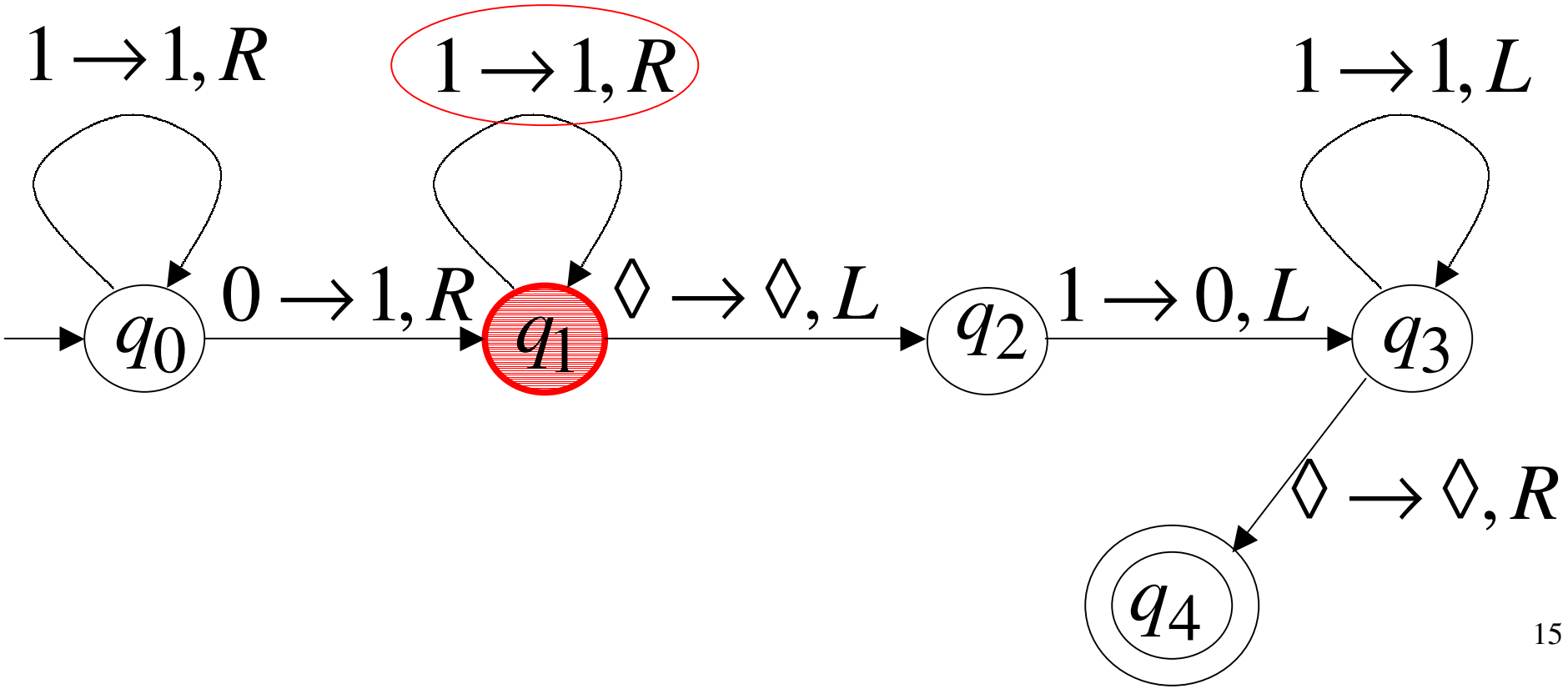
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 4


◊ 1 1 1 1 1 ◊
q_1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$1 \to 1, R$
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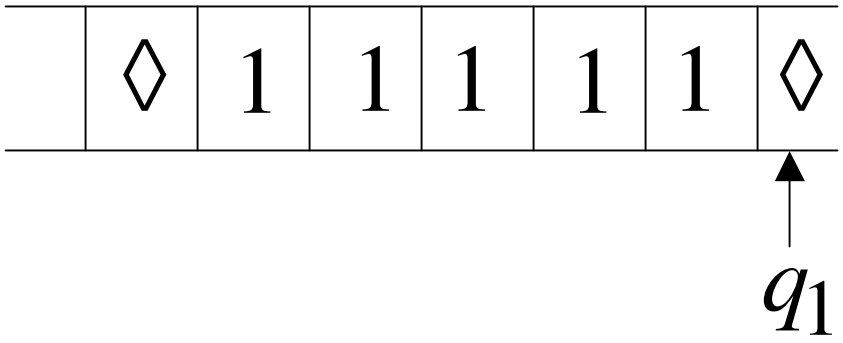
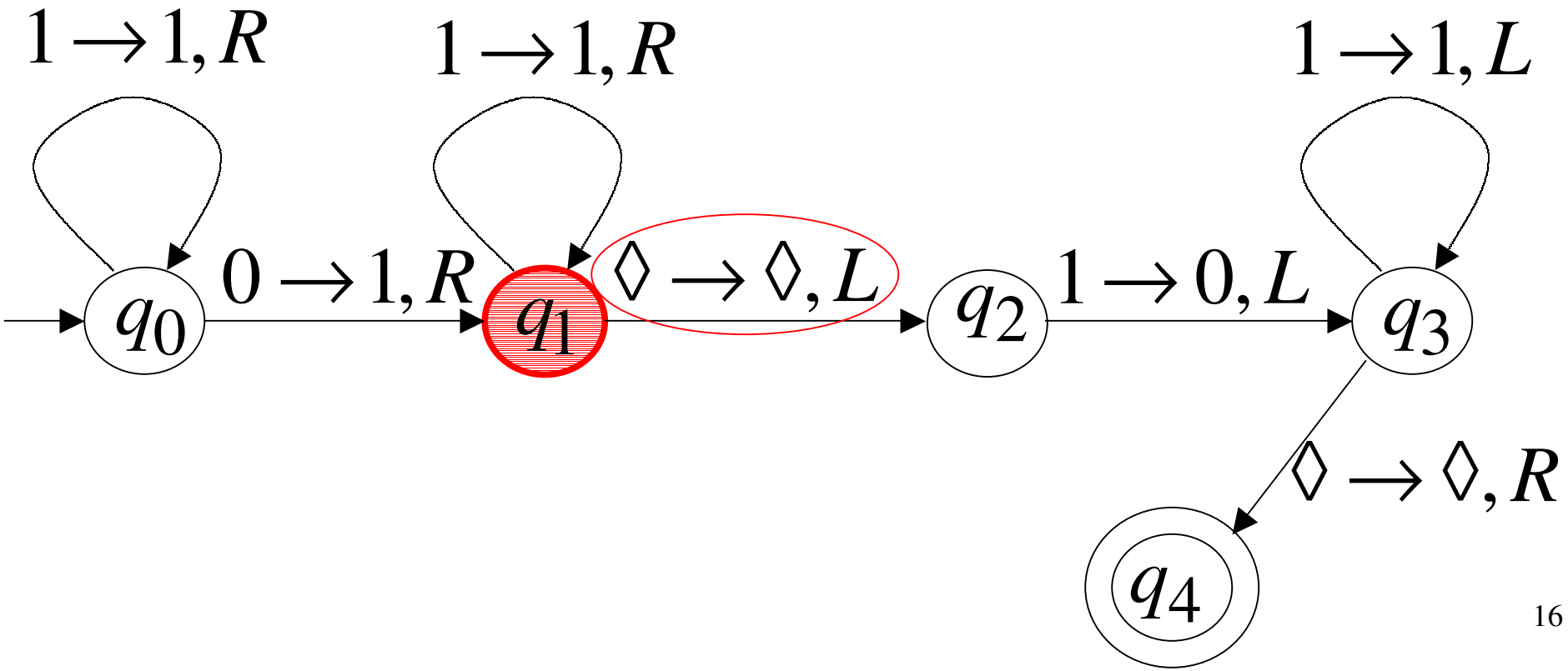
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 5


◊ 1 1 1 1 1 ◊
q_1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$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
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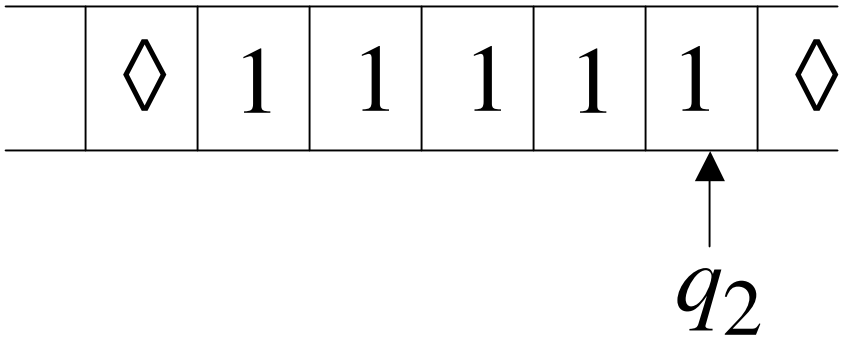
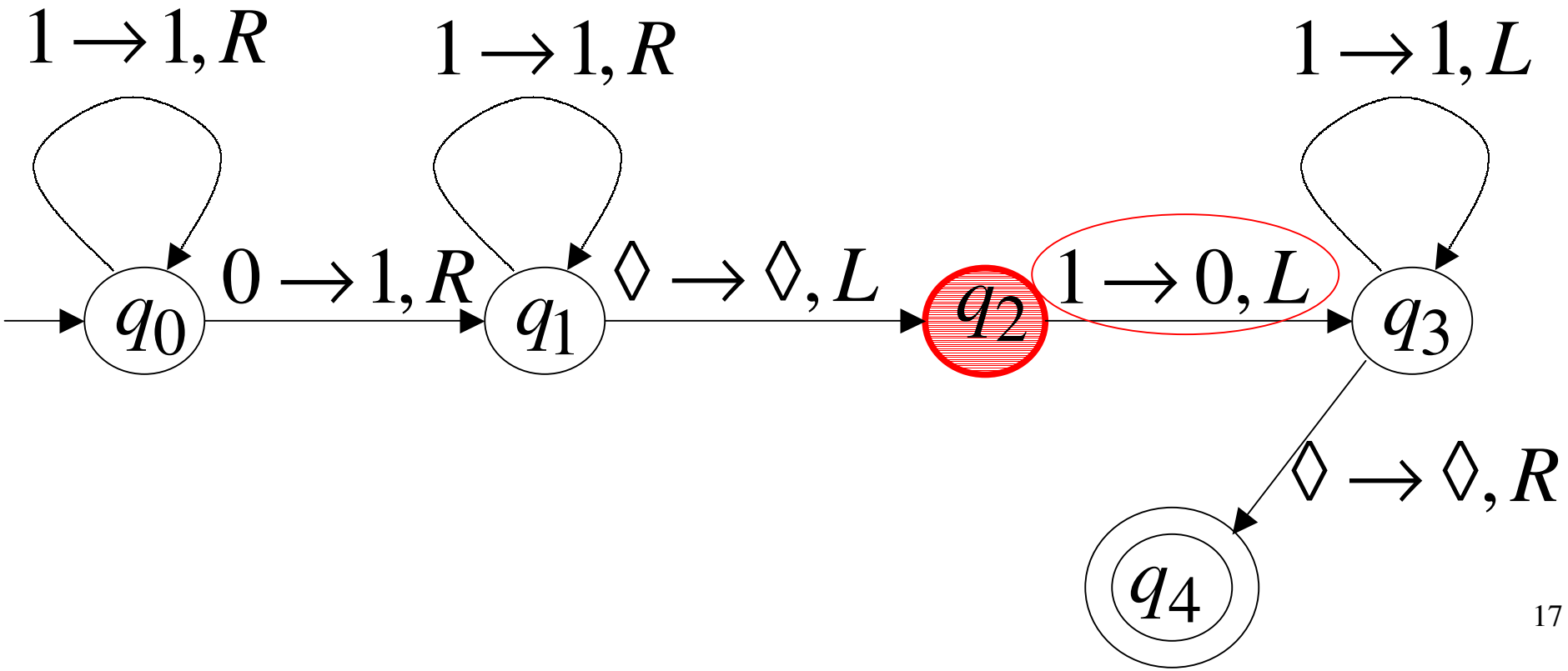
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 6


◊ 1 1 1 1 1 ◊
q_2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$1 \to 1, L$
$0 \to 1, R$
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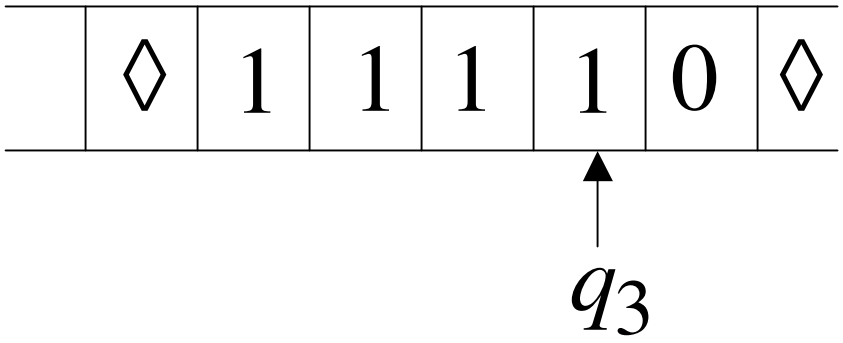
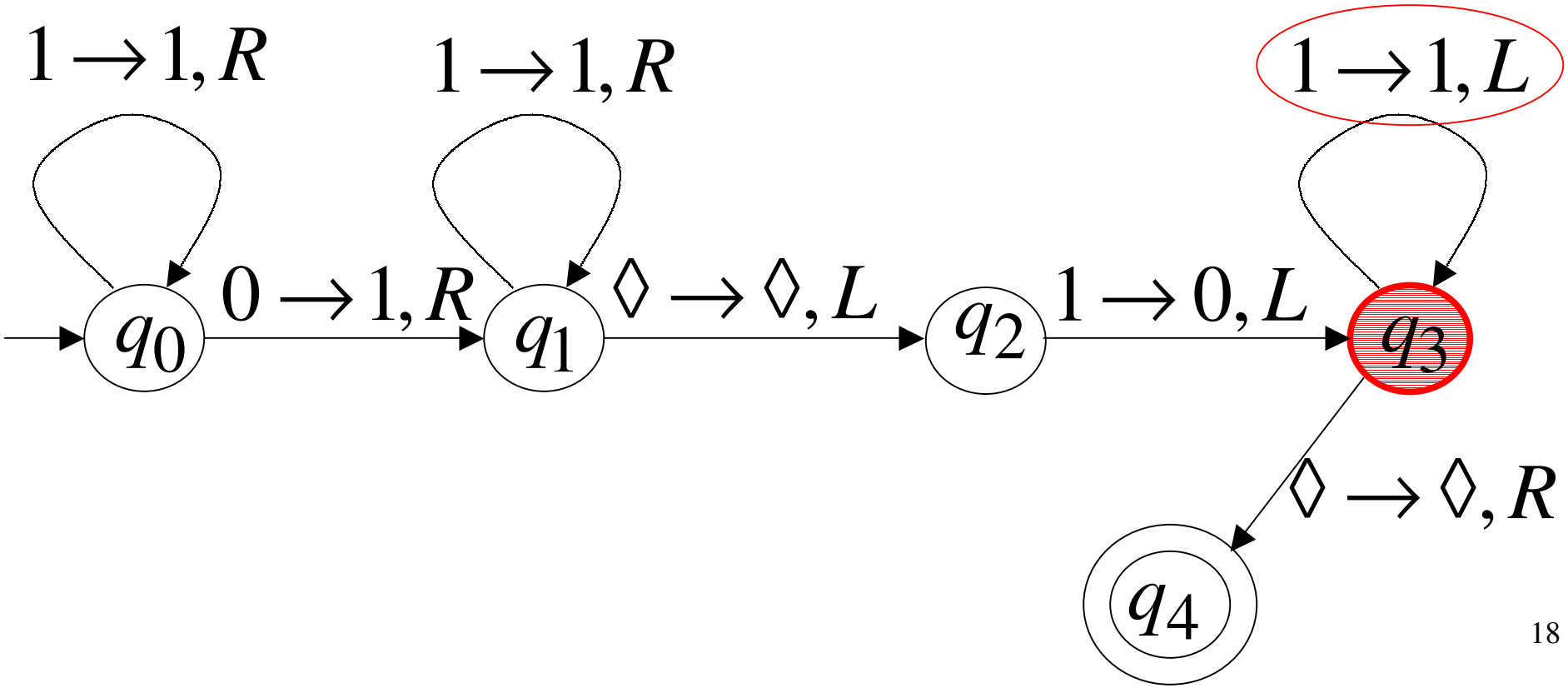
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 7


◊ 1 1 1 1 0 ◊
q_3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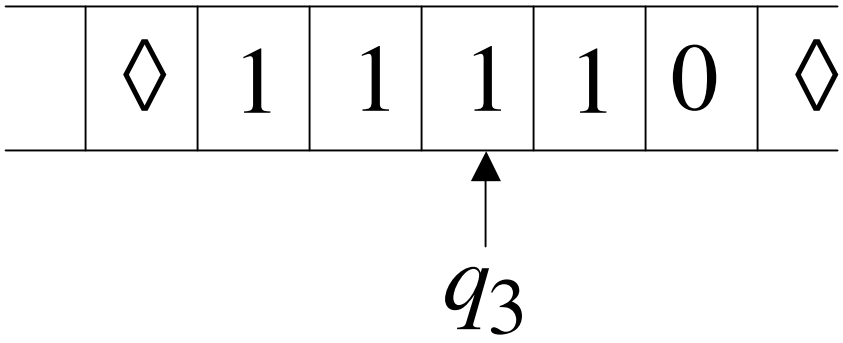
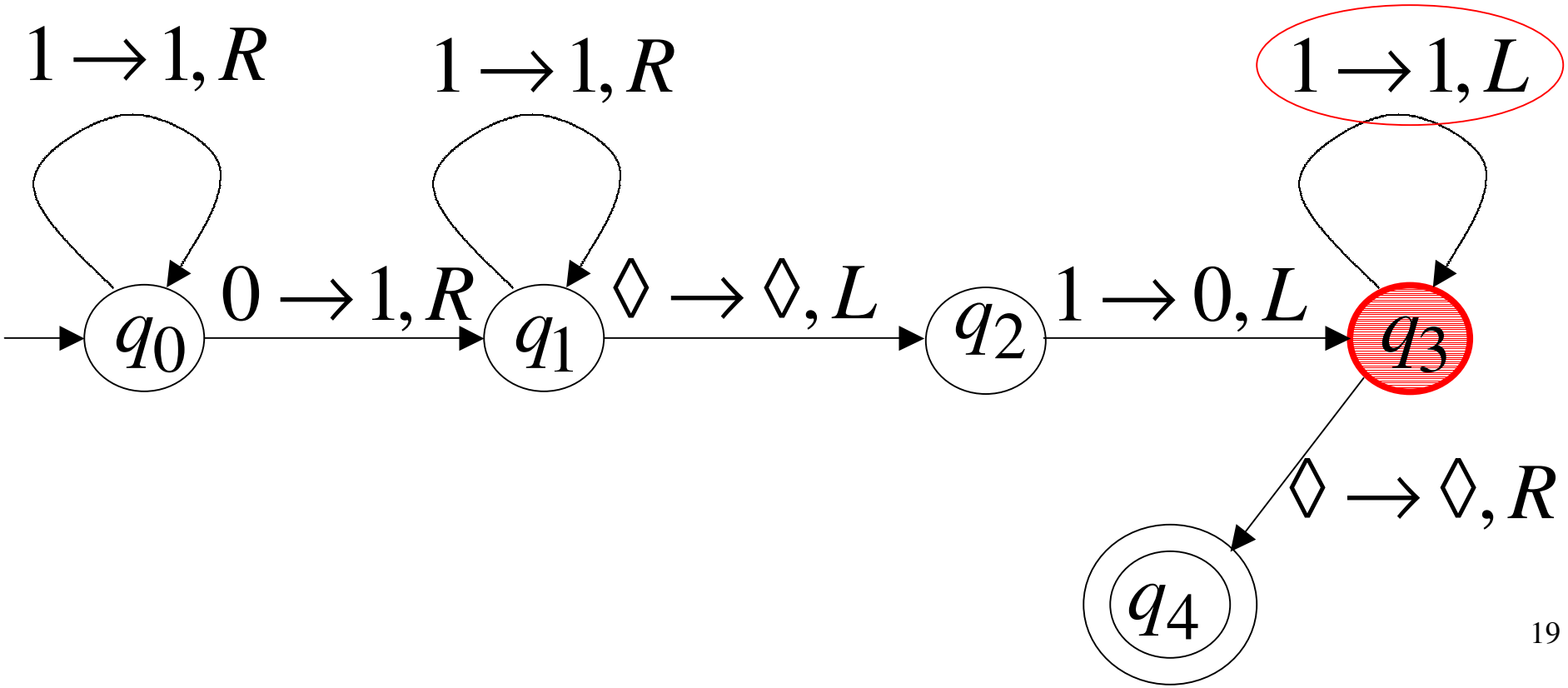
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 8


◊ 1 1 1 1 0 ◊
q_3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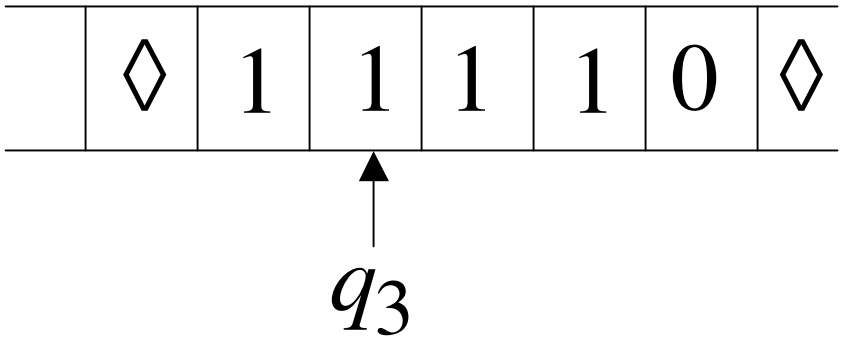
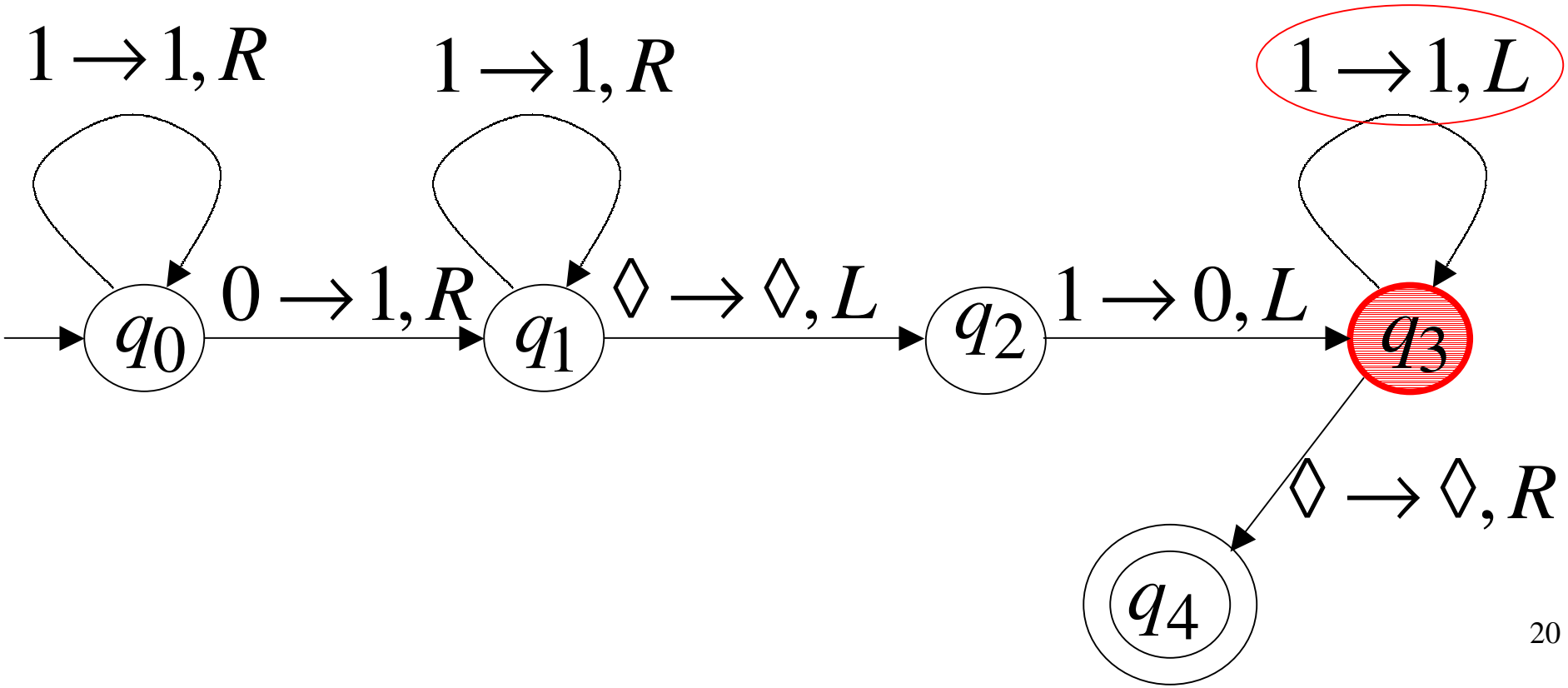
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 9


◊ 1 1 1 1 0 ◊
q_3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$1 \to 1, L$
$0 \to 1, R$
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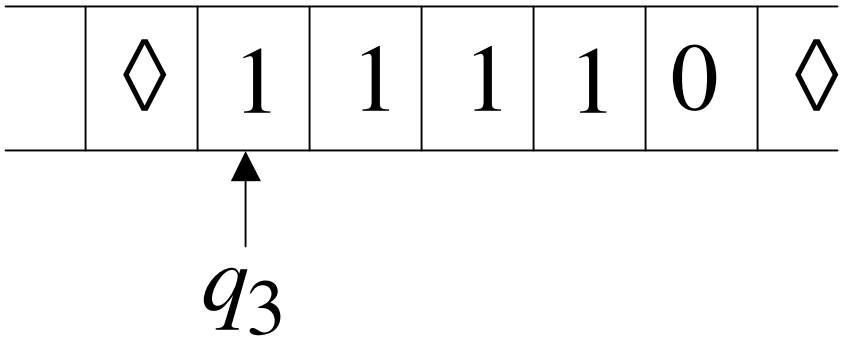
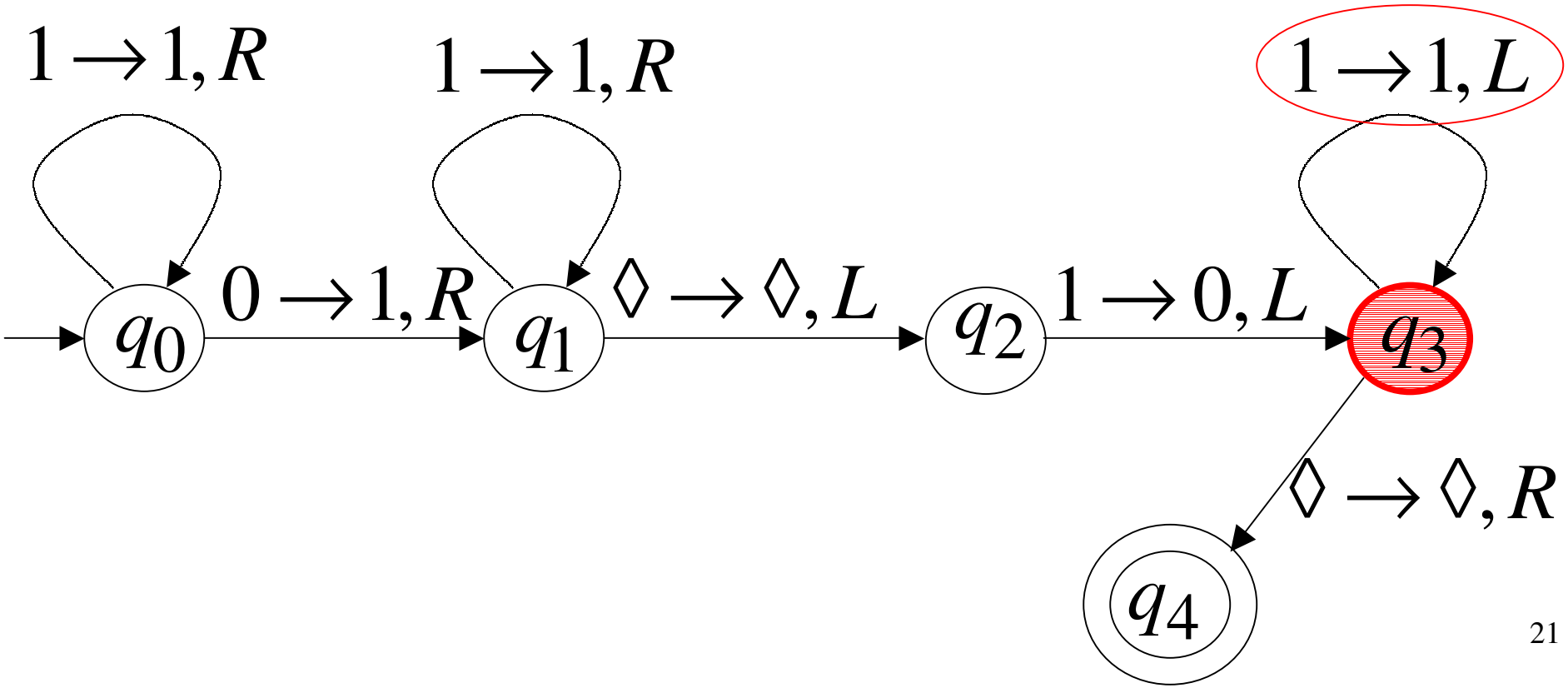
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 10


◊ 1 1 1 1 0 ◊
q_3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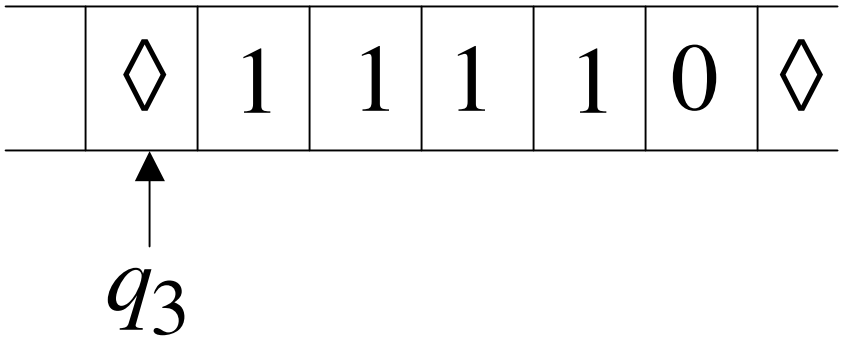
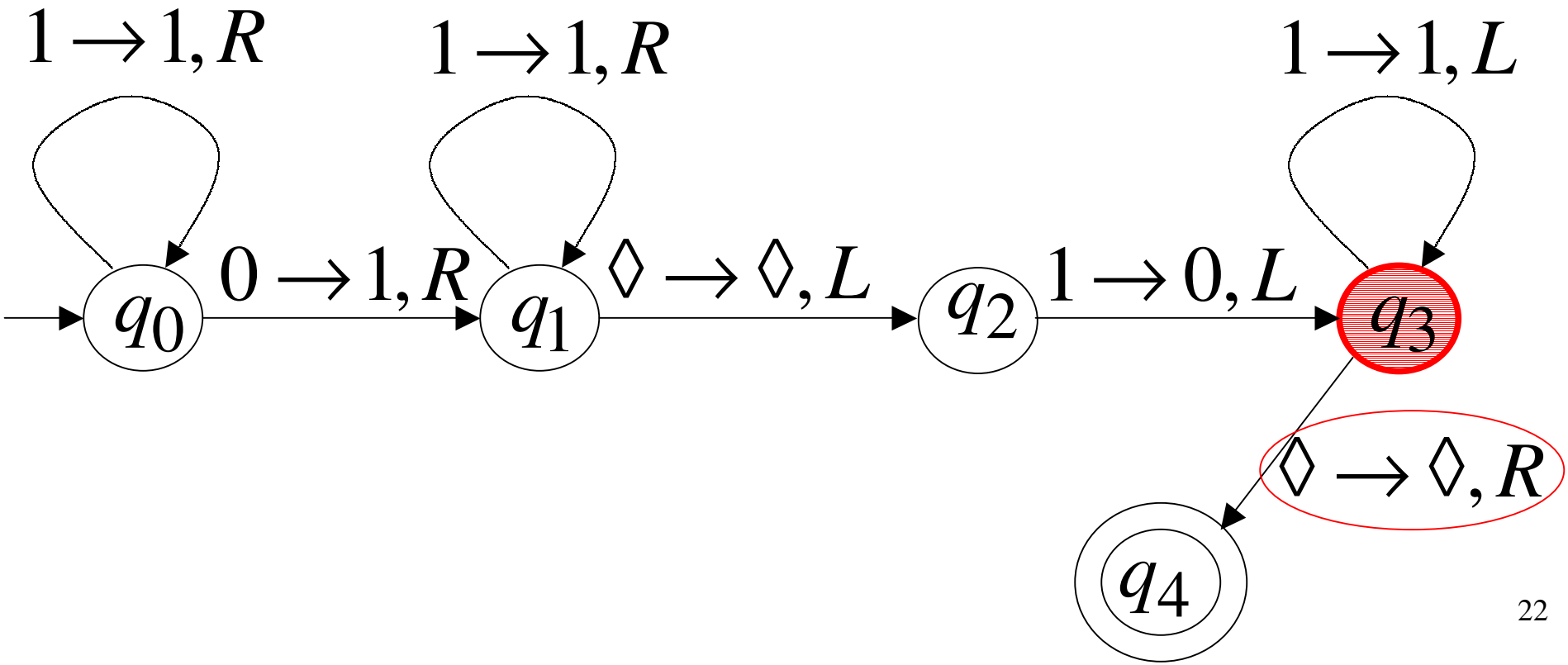
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 11


◊ 1 1 1 1 0 ◊
q_3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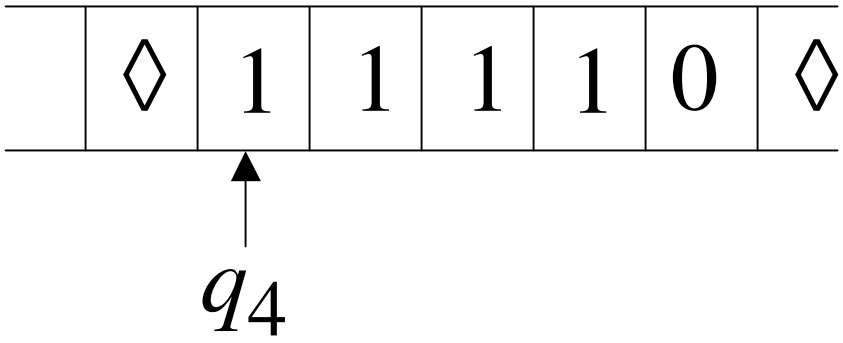
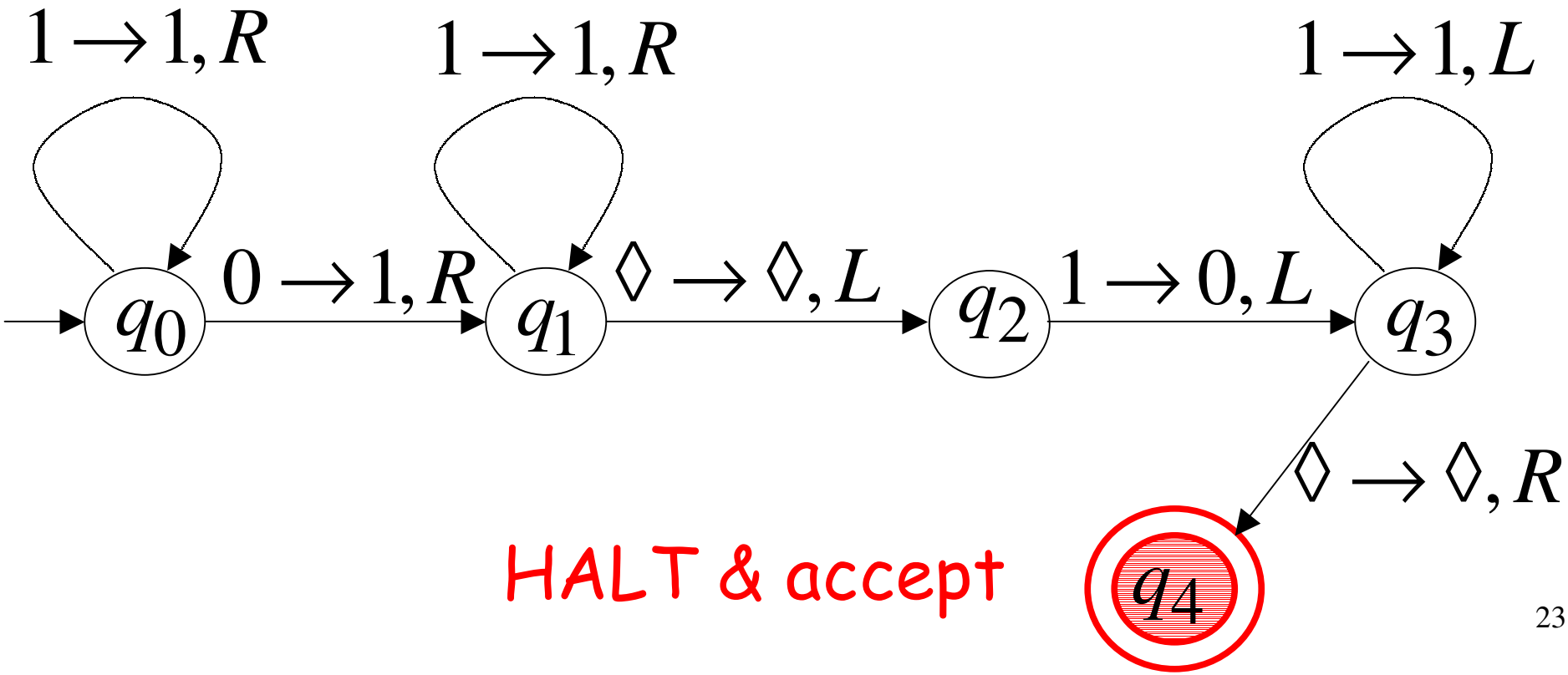
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 12


◊ 1 1 1 1 0 ◊
q_4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$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4
HALT & accept

Another Example

The function $f(x) = 2x$ is computable

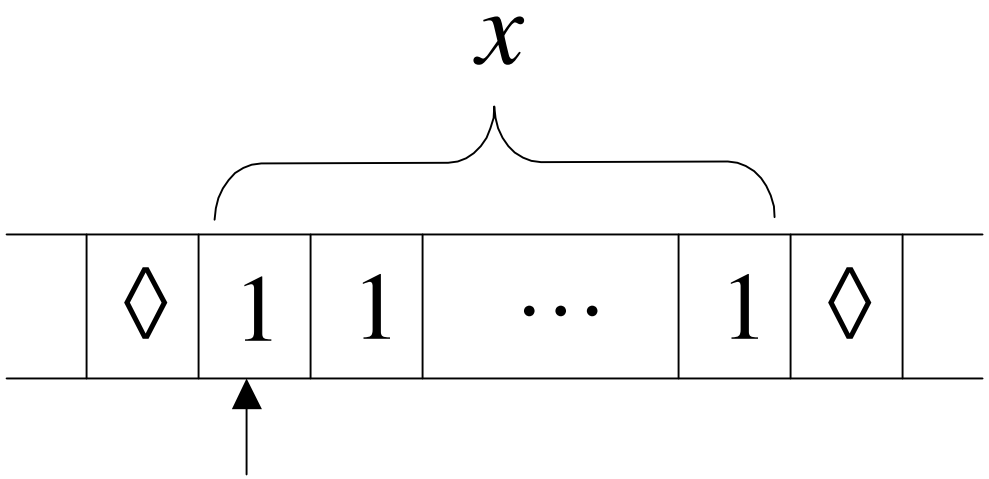
x is integer

Turing Machine:

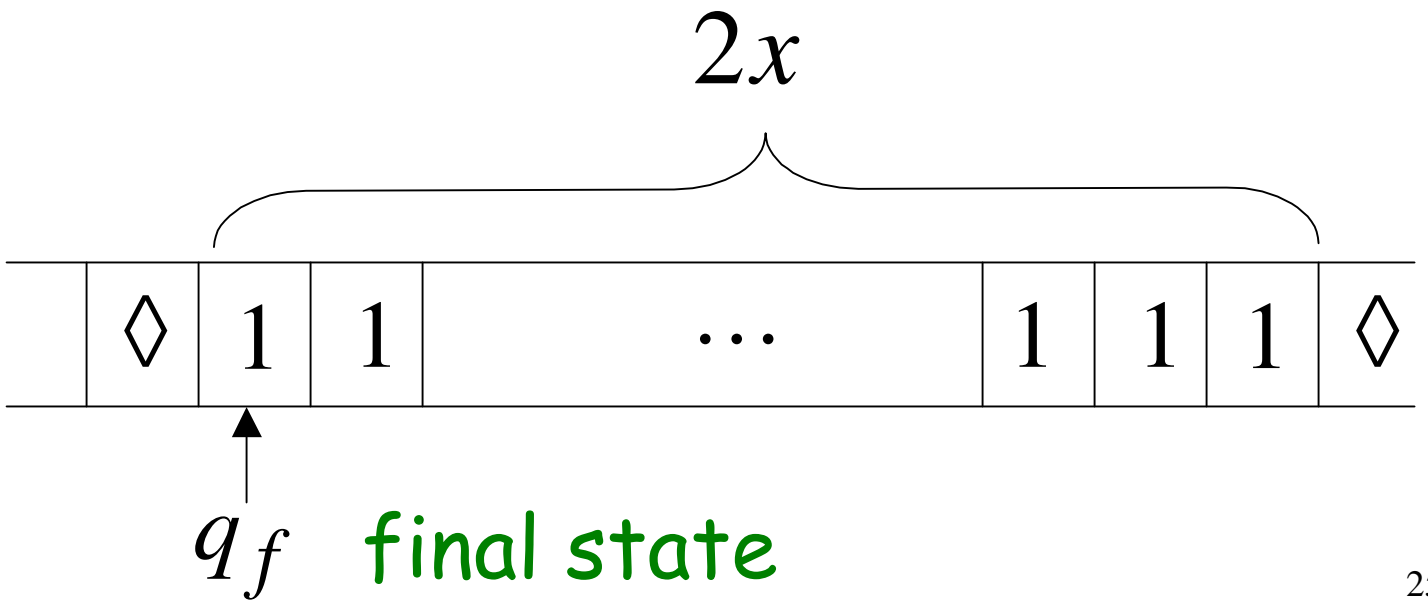
Input string: x unary

Output string: xx unary

Start



Finish

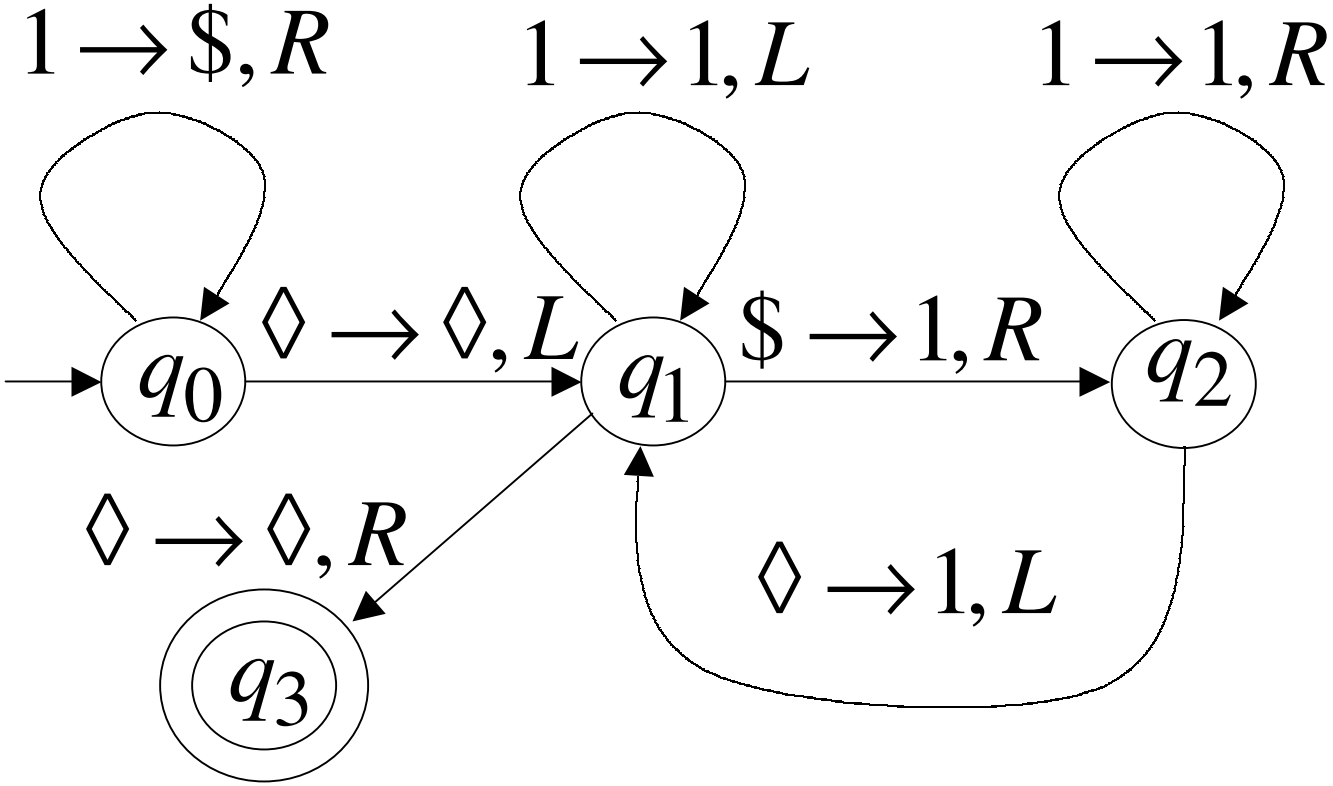


Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with $
- Repeat:
  - Find rightmost $, replace it with 1
  - Go to right end, insert 1

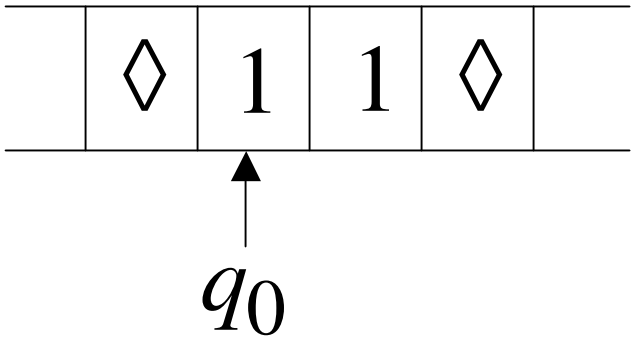
Until no more $ remain

Turing Machine for $f(x) = 2x$

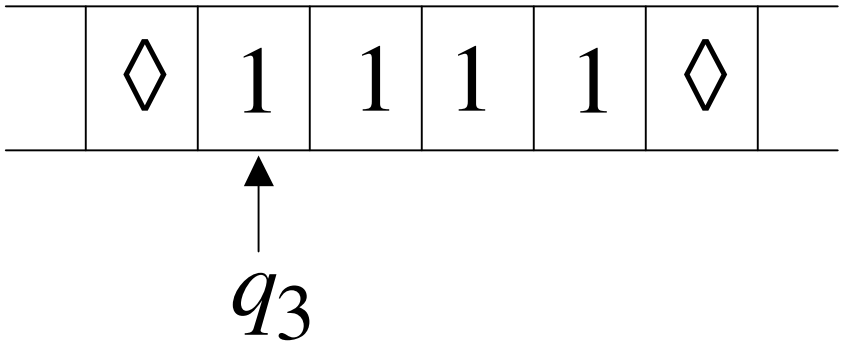


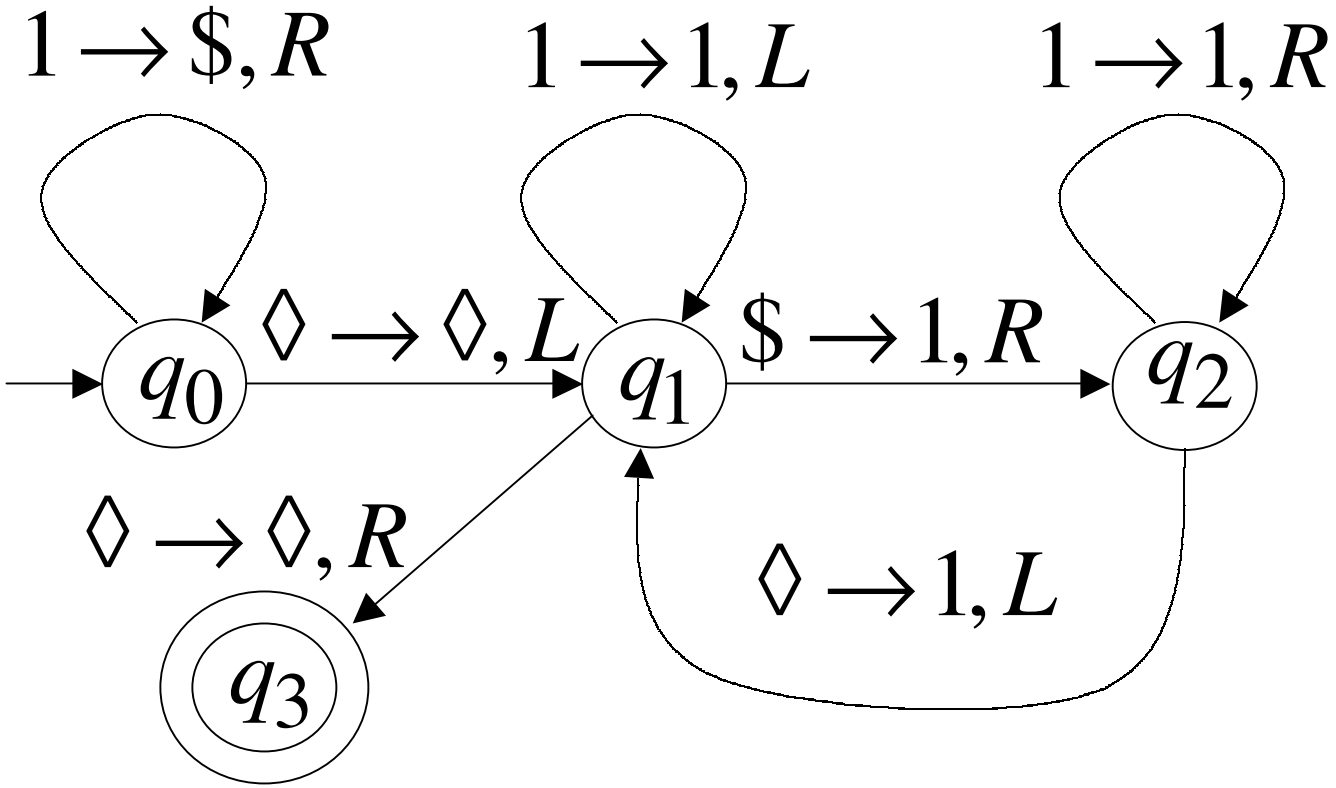
Example

Start



Finish





Another Example

The function

is computable

$$f(x,y)=\begin{cases}1 & \text{if } x>y\\ 0 & \text{if } x\leq y\end{cases}$$

Turing Machine for

$$f(x,y)=\begin{cases}1 & \text{if } x>y\\ 0 & \text{if } x\le y\end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched

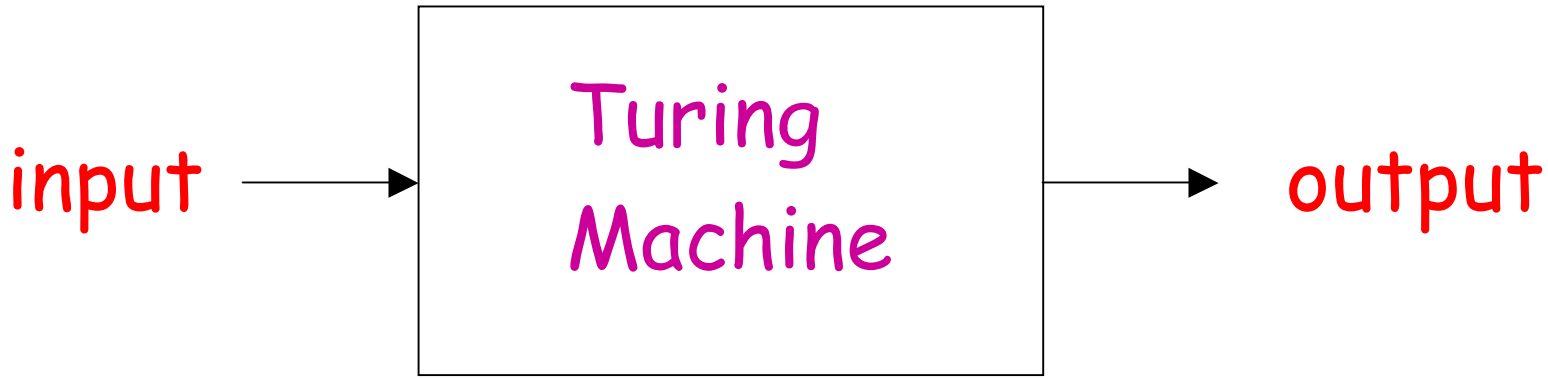
  erase tape, write 1

  else

  erase tape, write 0

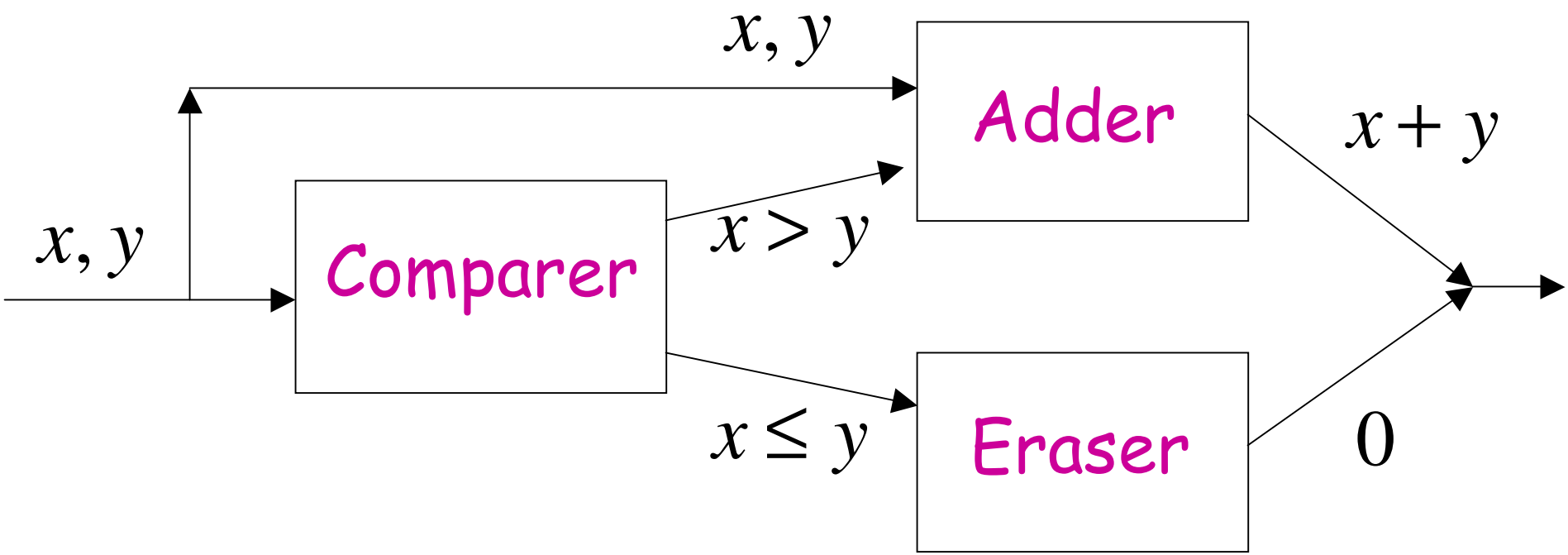
Combining Turing Machines

Block Diagram



Example:

$$f(x,y)=\begin{cases} x+y & \text{if } x>y \\ 0 & \text{if } x\le y \end{cases}$$



Turing's Thesis

Question: Do Turing machines have the same power with a digital computer?

Intuitive answer: Yes

There is no formal answer!!!

Turing's thesis:

Any computation carried out
by mechanical means
can be performed by a Turing Machine

(1930)

Computer Science Law:

A computation is mechanical
if and only if
it can be performed by a Turing Machine

There is no known model of computation
more powerful than Turing Machines

Definition of Algorithm:

An algorithm for function $f(w)$

is a

Turing Machine which computes $f(w)$

Algorithms are Turing Machines

When we say:

There exists an algorithm

We mean:

There exists a Turing Machine