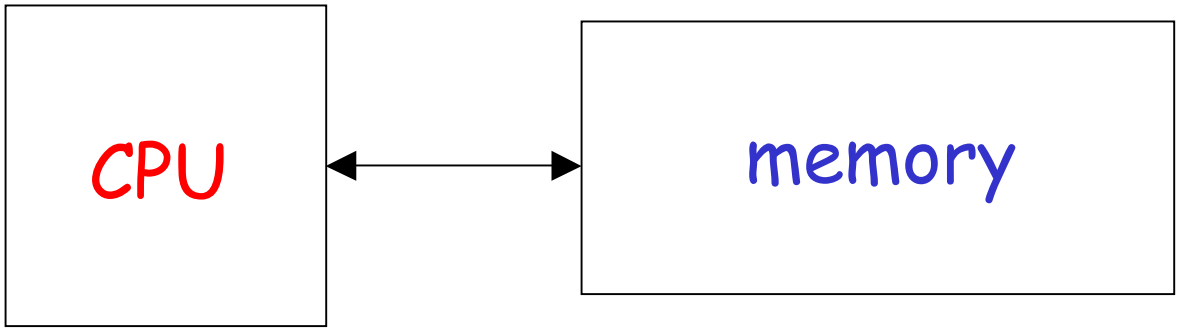
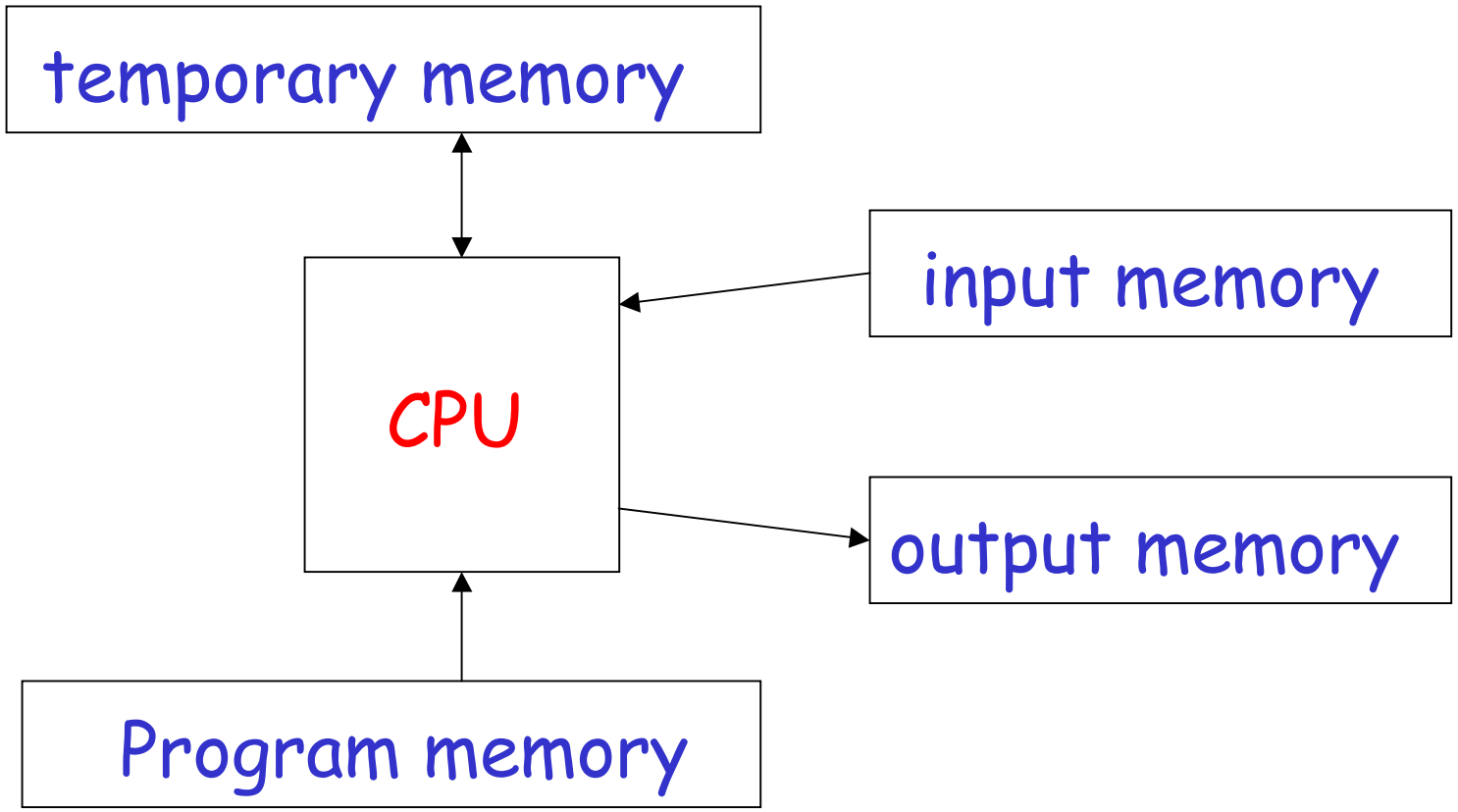
CSCI-2400

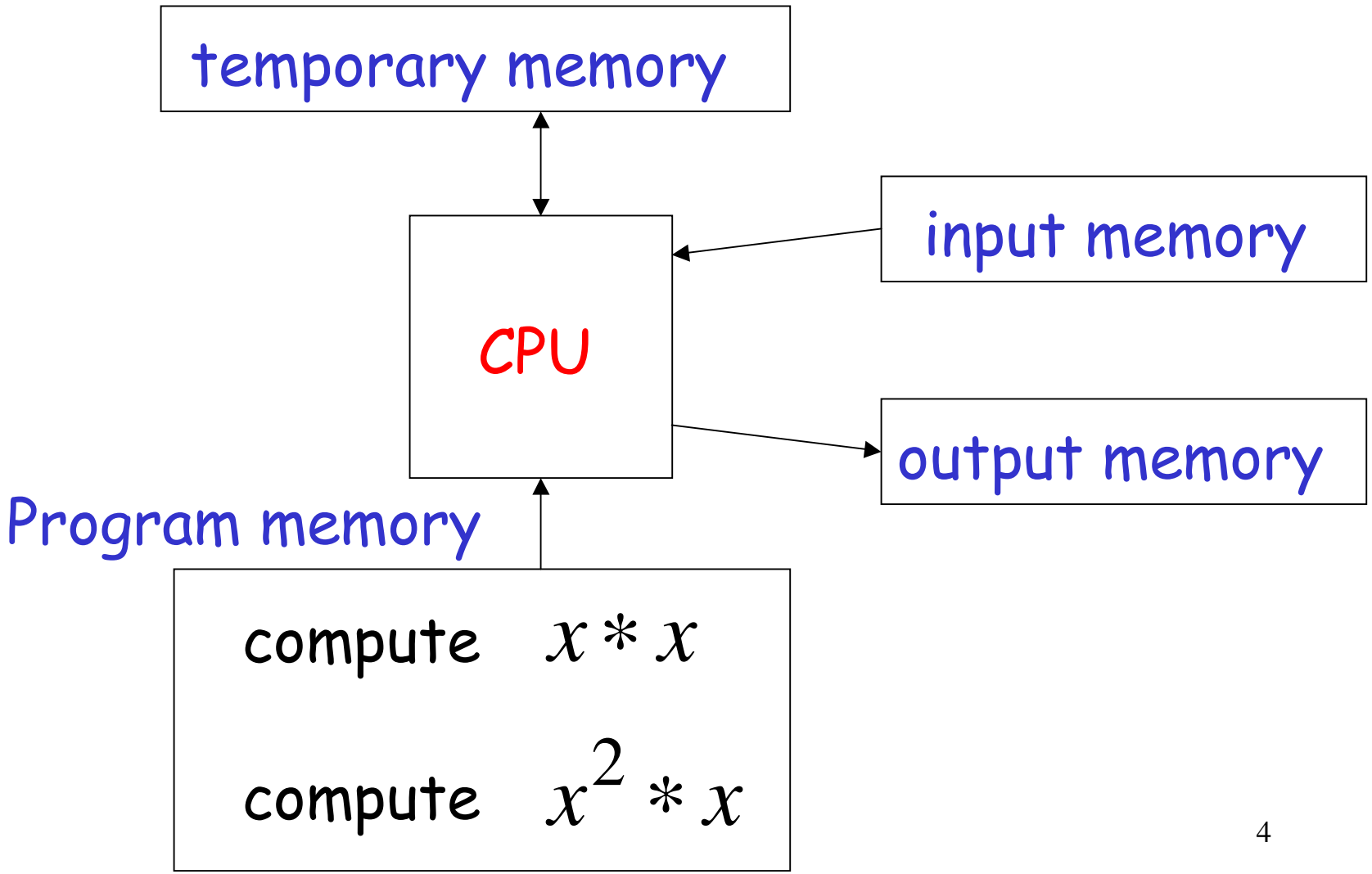
Models of Computation

Computation

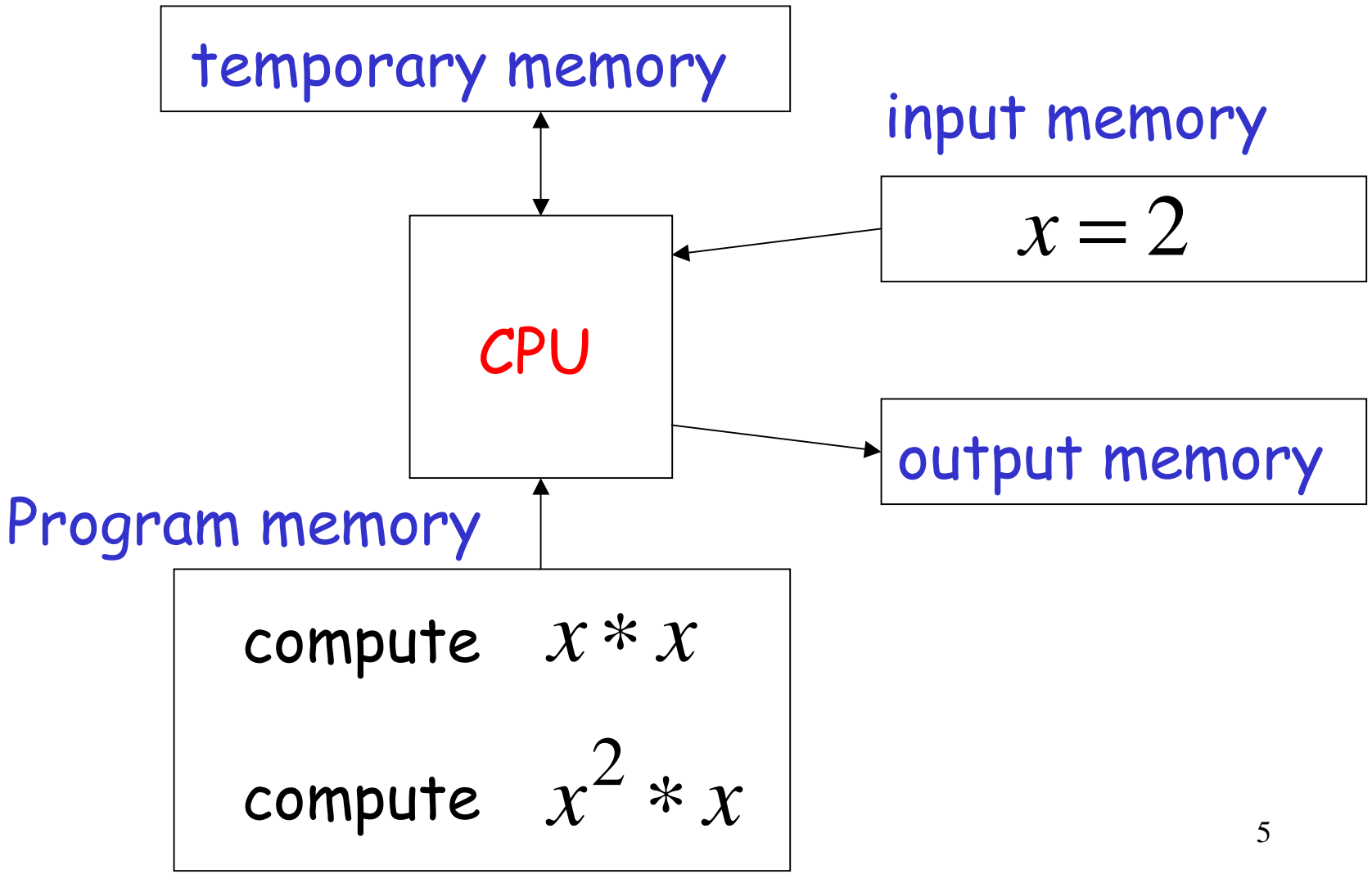



temporary memory
CPU
input memory
output memory
Program memory

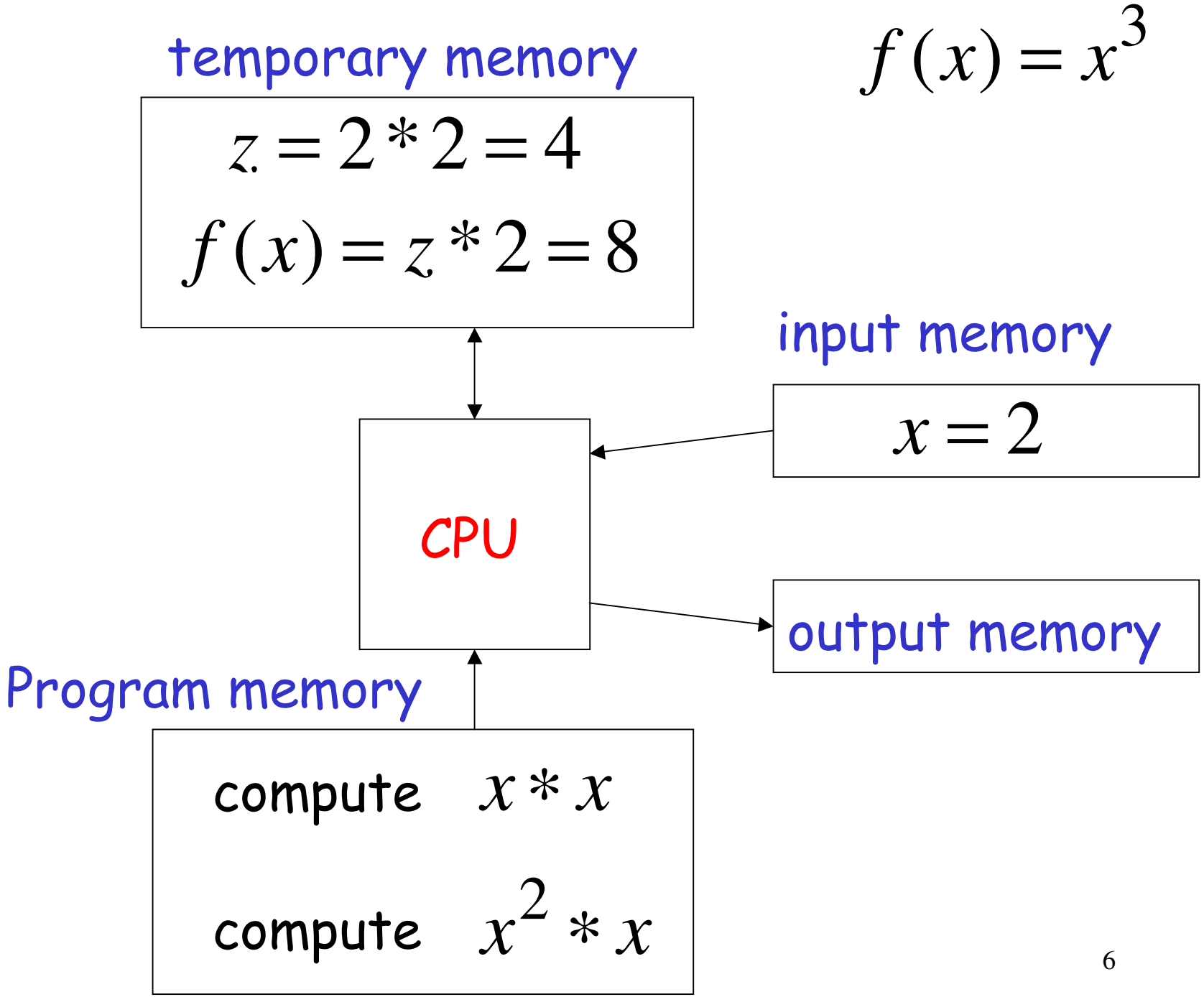
Example: $f(x) = x^3$



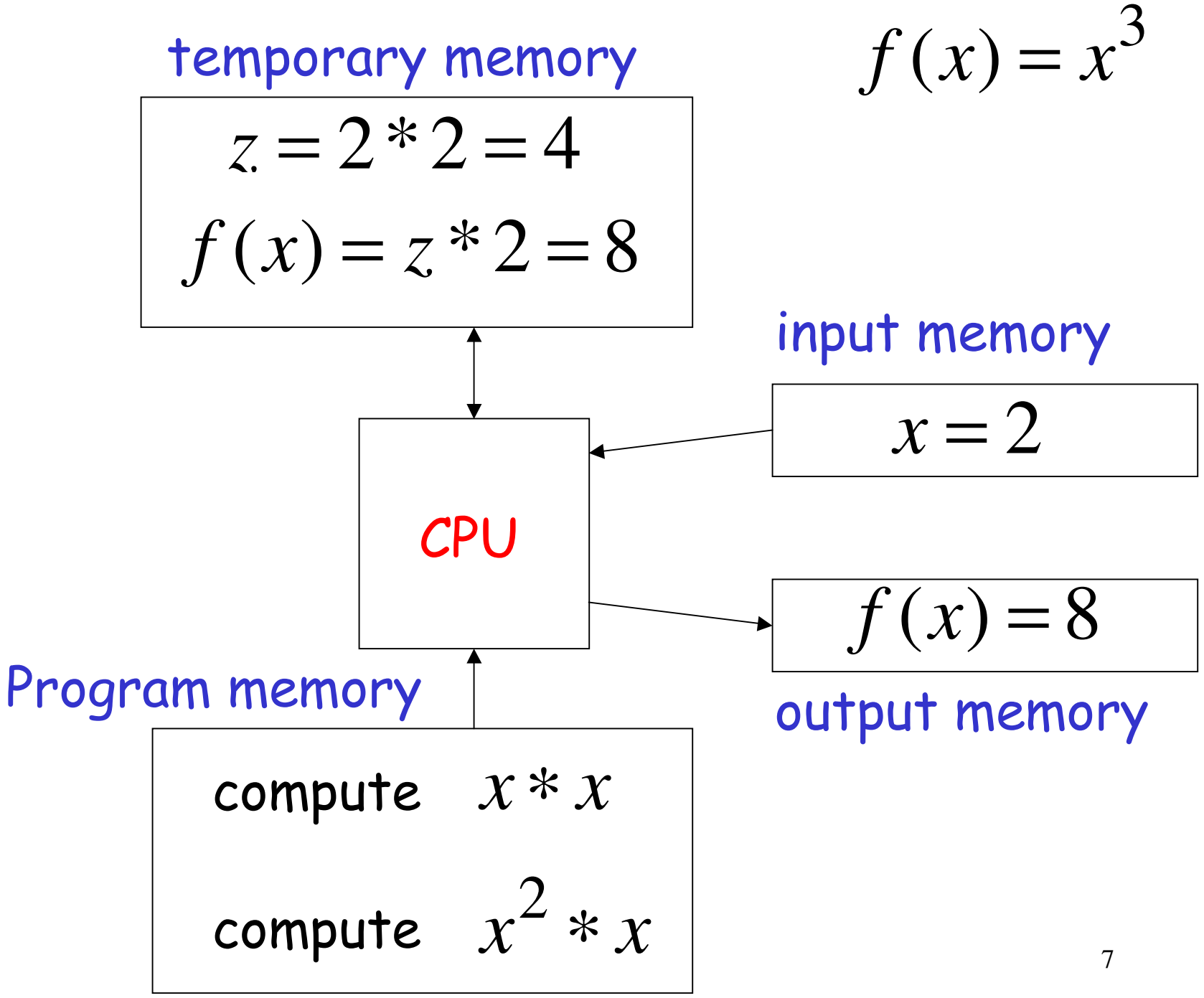
$$f(x) = x^3$$


temporary memory
input memory
$x = 2$
CPU
output memory
Program memory
compute $x * x$
compute $x^2 * x$

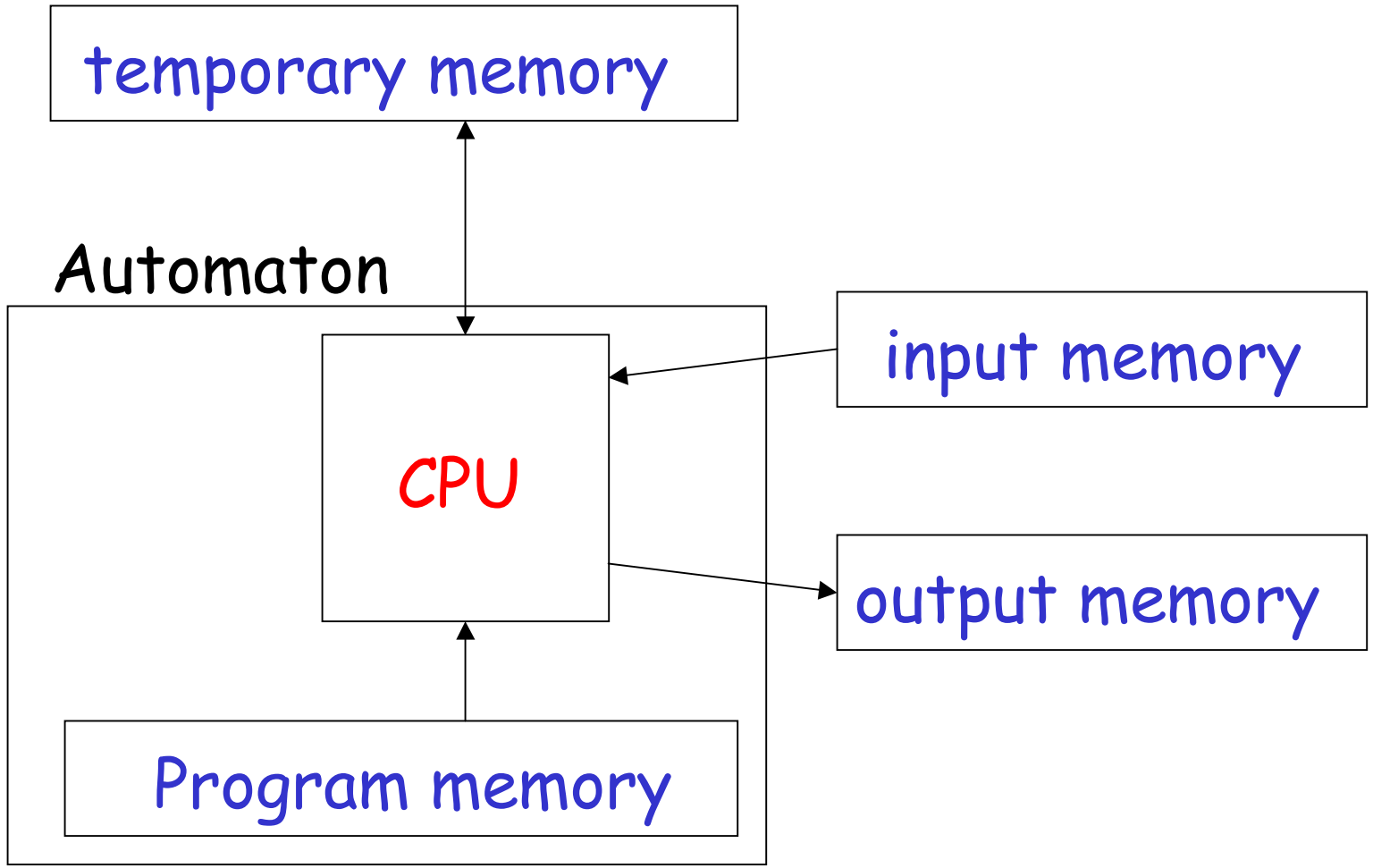
$$f(x) = x^3$$


temporary memory
$z = 2 * 2 = 4$
$f(x) = z * 2 = 8$
input memory
$x = 2$
CPU
output memory
Program memory
compute $x * x$
compute $x^2 * x$

$$f(x) = x^3$$


temporary memory
$z = 2 * 2 = 4$
$f(x) = z * 2 = 8$
input memory
$x = 2$
CPU
$f(x) = 8$
output memory
Program memory
compute $x * x$
compute $x^2 * x$

Automaton

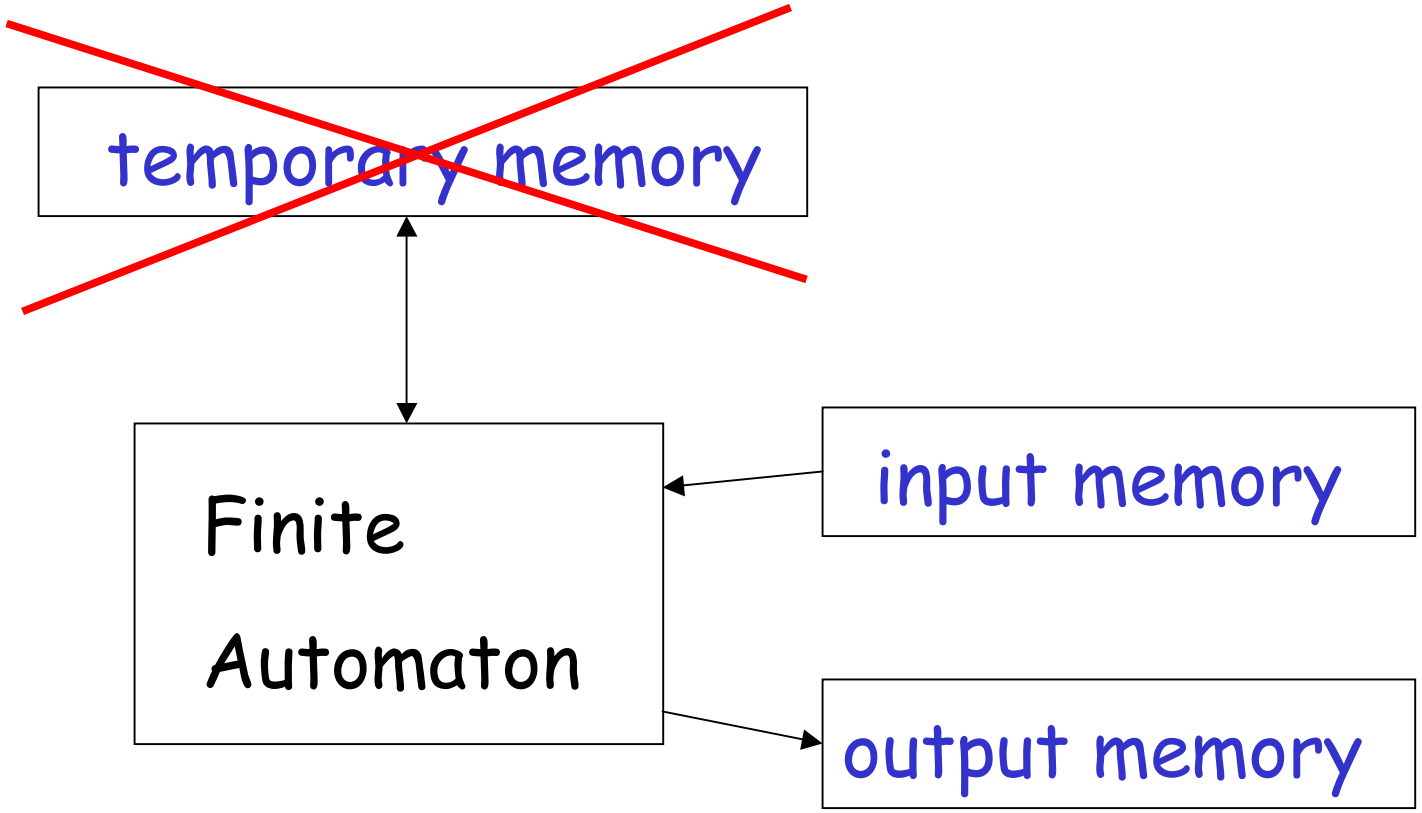

temporary memory
Automaton
CPU
input memory
output memory
Program memory

Different Kinds of Automata

Automata are distinguished by the temporary memory

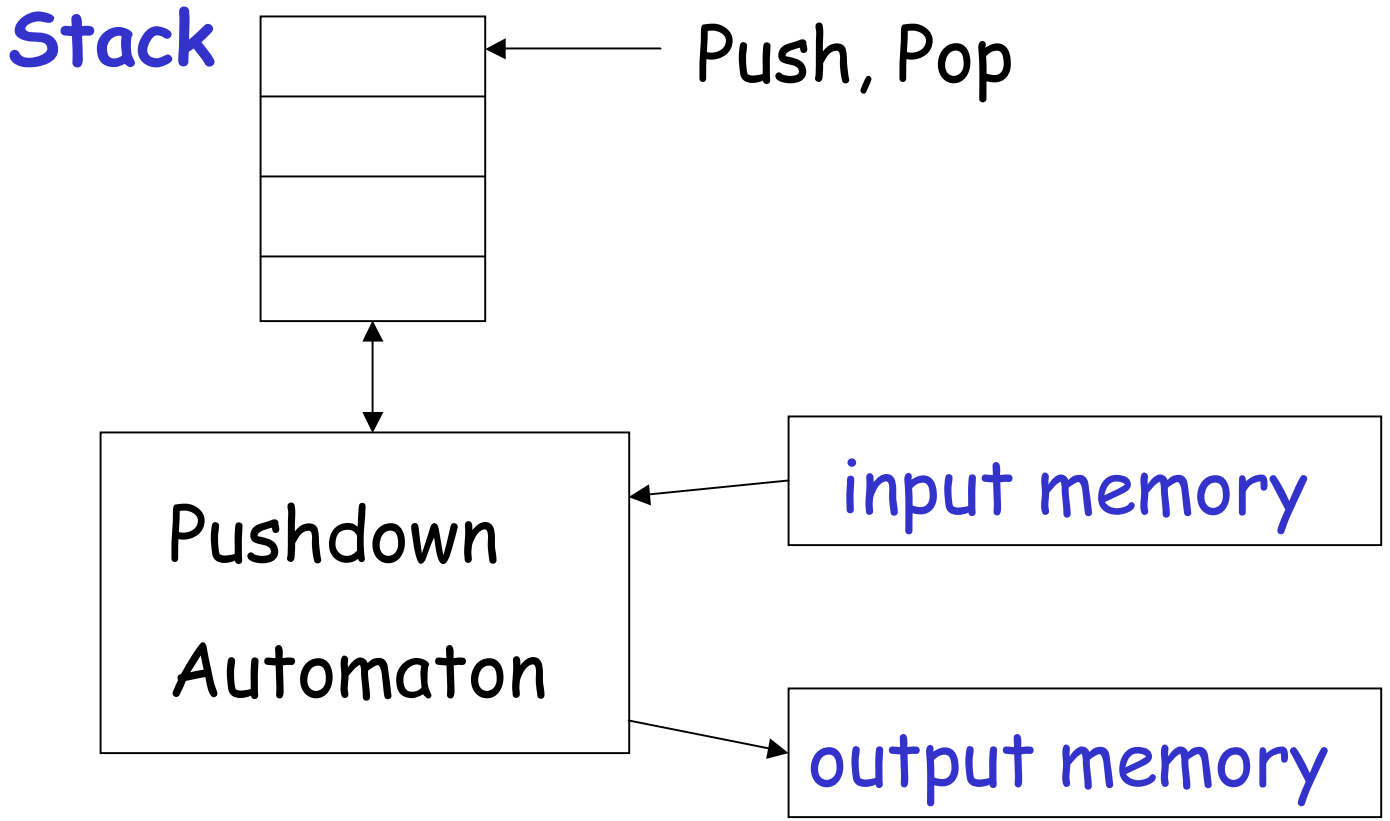
- **Finite Automata**: no temporary memory
- **Pushdown Automata**: stack
- **Turing Machines**: random access memory

Finite Automaton



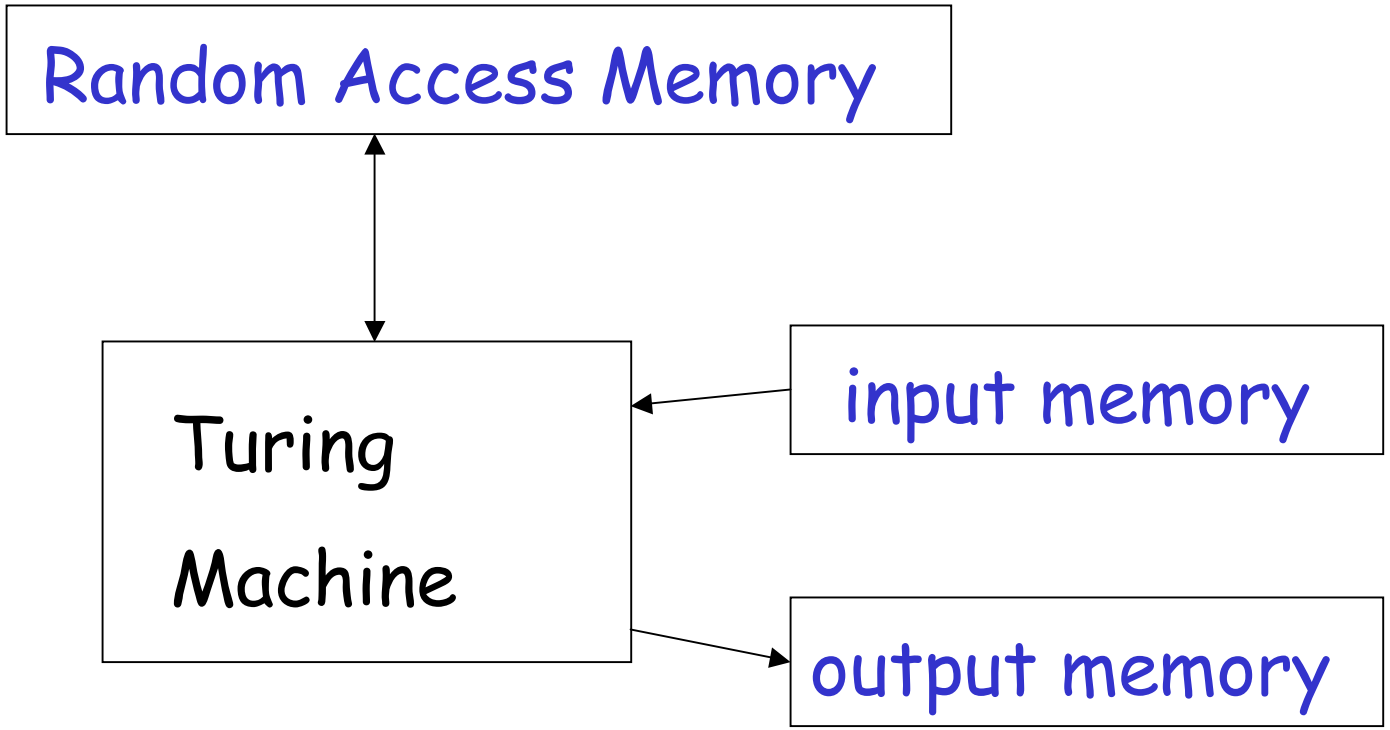
Vending Machines (small computing power)

Pushdown Automaton



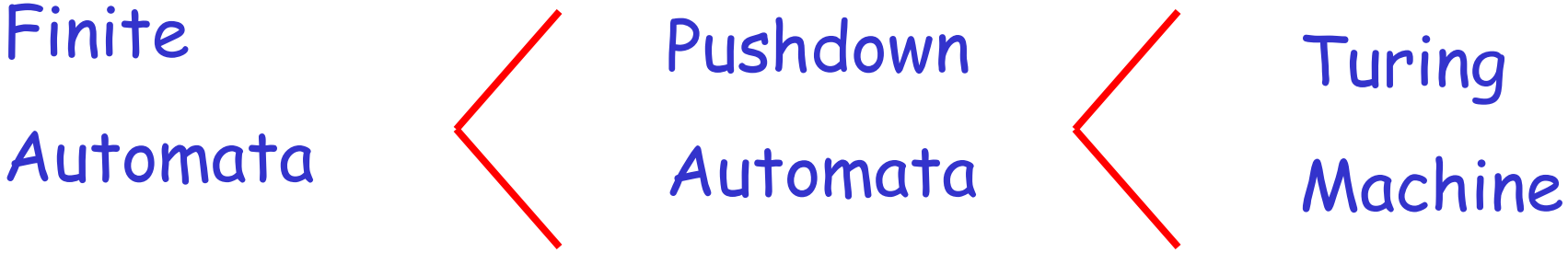
Programming Languages (medium computing power)

Turing Machine



Algorithms (highest computing power)

Power of Automata



We will show later in class

- How to build compilers for programming languages
- Some computational problems cannot be solved
- Some problems are hard to solve

Mathematical Preliminaries

Mathematical Preliminaries

- Sets
- Functions
- Relations
- Graphs
- Proof Techniques

SETS

A set is a collection of elements

$$A = \{1, 2, 3\}$$

$$B = \{train, bus, bicycle, airplane\}$$

We write

$$1 \in A$$

$$ship \notin B$$

Set Representations

$C = \{ a, b, c, d, e, f, g, h, i, j, k \}$

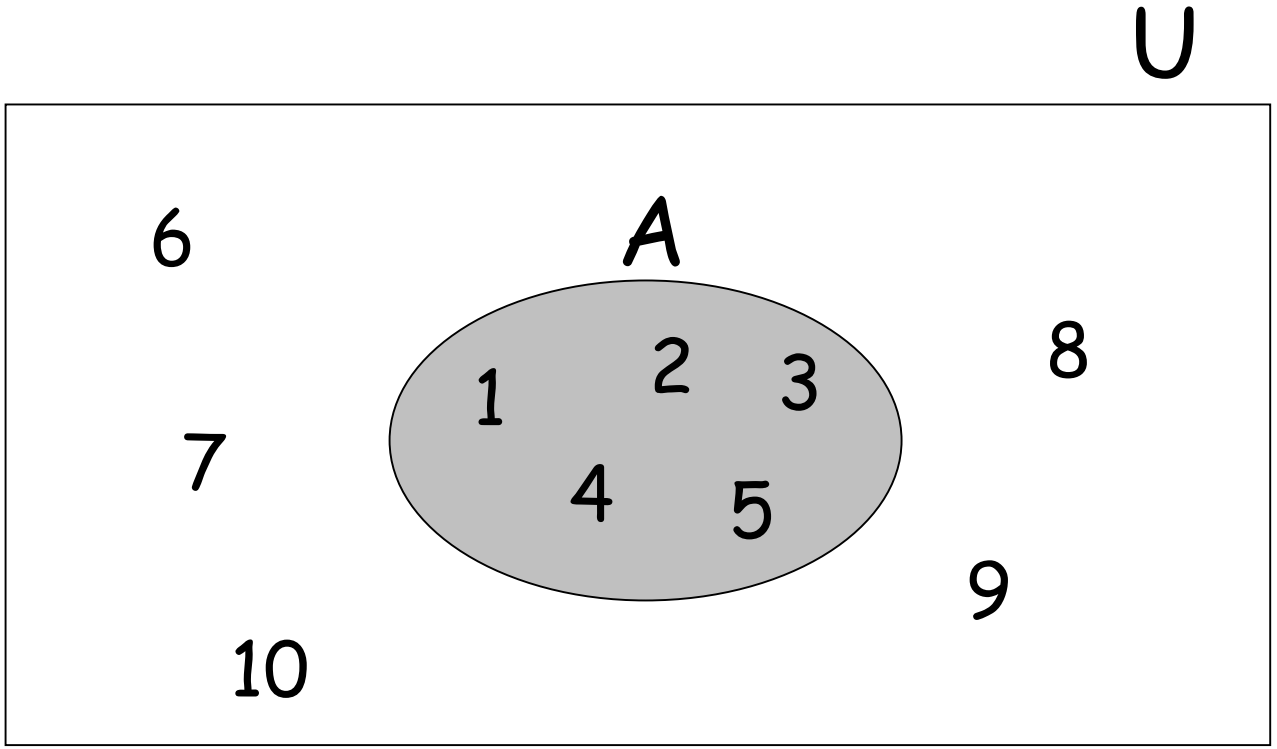
$C = \{ a, b, \ldots, k \}$ ⟶ *finite set*

$S = \{ 2, 4, 6, \ldots \}$ ⟶ *infinite set*

$S = \{ j : j > 0, \text{ and } j = 2k \text{ for some } k>0 \}$

$S = \{ j : j \text{ is nonnegative and even} \}$

$A = \{ 1, 2, 3, 4, 5 \}$



Universal Set: All possible elements

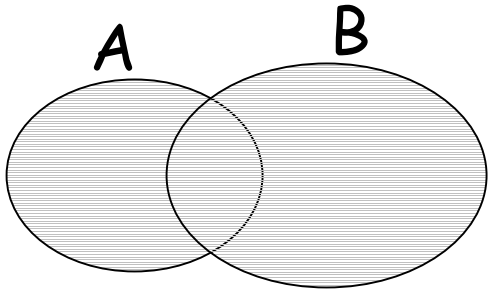
$U = \{ 1 , \ldots , 10 \}$

Set Operations

$A = \{ 1, 2, 3 \}$ $B = \{ 2, 3, 4, 5\}$

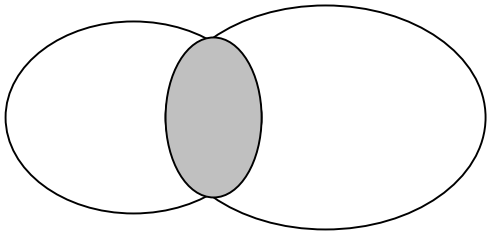
- Union

$A \cup B = \{ 1, 2, 3, 4, 5 \}$



- Intersection

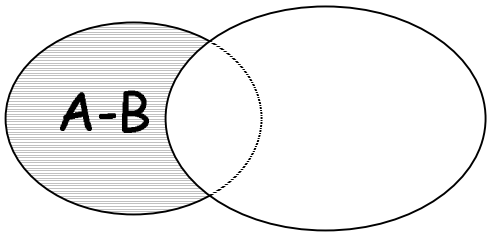
$A \cap B = \{ 2, 3 \}$



- Difference

$A - B = \{ 1 \}$

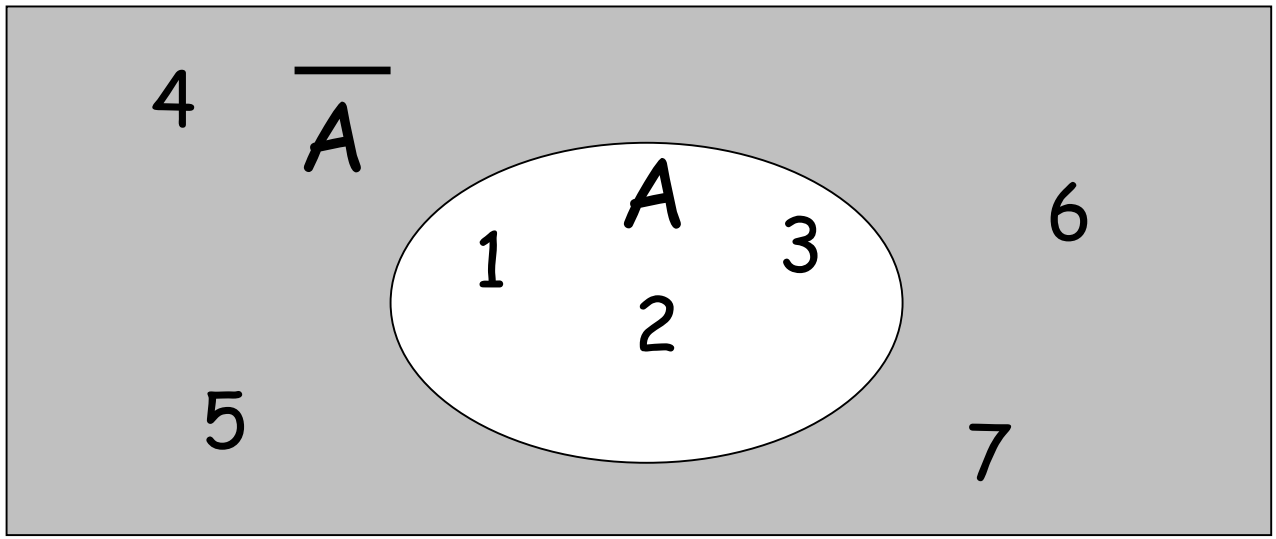
$B - A = \{ 4, 5 \}$



- Complement

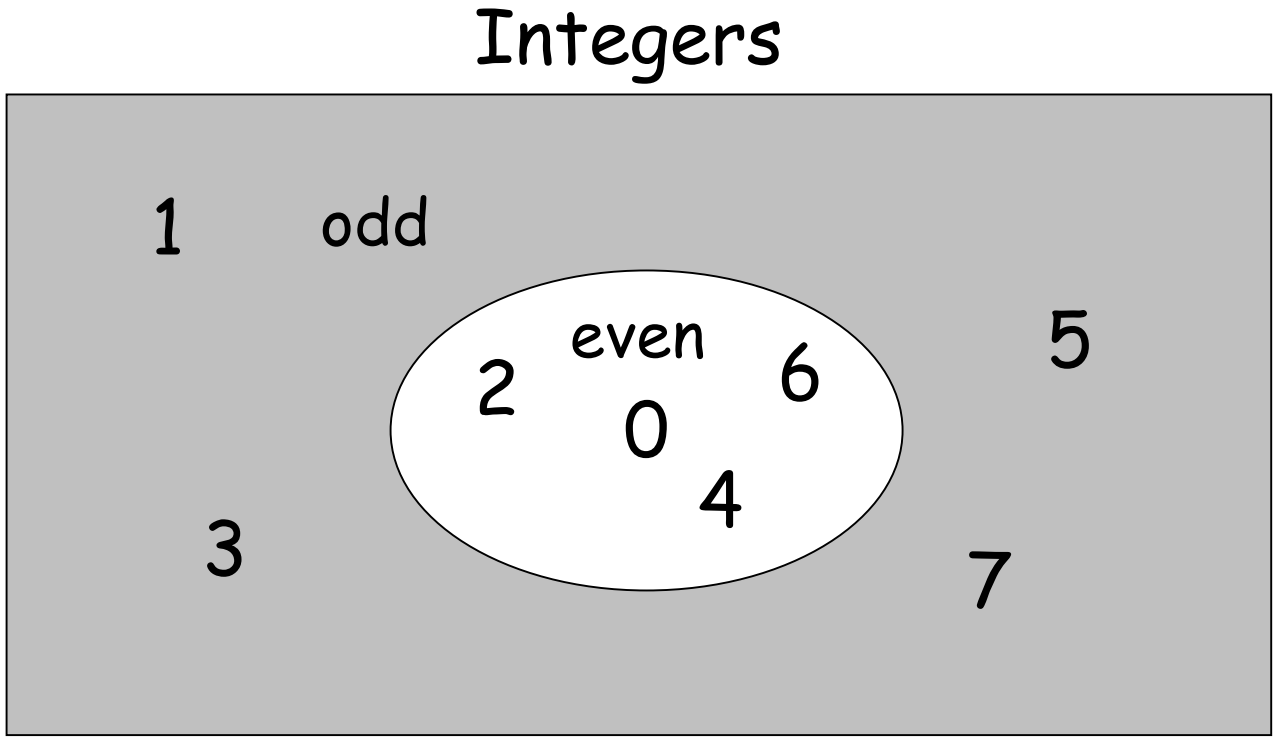
Universal set = {1, …, 7}

$A = \{ 1, 2, 3 \}$ ⇒ $\overline{A} = \{ 4, 5, 6, 7 \}$



$\overline{\overline{A}} = A$

$\overline{\{ \text{even integers} \}} = \{ \text{odd integers} \}$



DeMorgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Empty, Null Set: $\emptyset$

$\emptyset = \{\}$

$S \cup \emptyset = S$

$S \cap \emptyset = \emptyset$

$S - \emptyset = S$

$\emptyset - S = \emptyset$

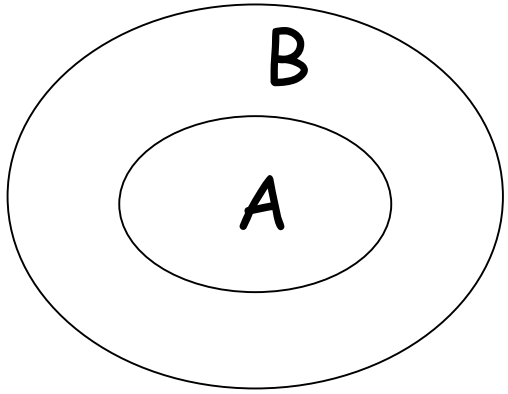
$\overline{\emptyset}$ = Universal Set

Subset

$A = \{ 1, 2, 3\}$ $B = \{ 1, 2, 3, 4, 5 \}$

$A \subseteq B$

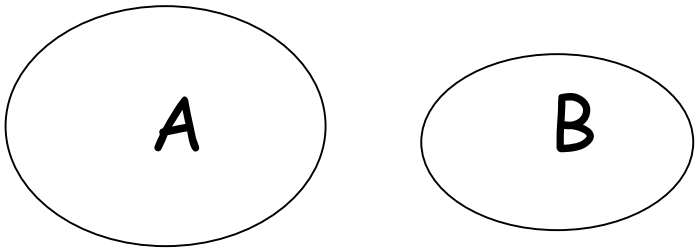
Proper Subset: $A \subset B$



Disjoint Sets

$A = \{ 1, 2, 3 \}$ $B = \{ 5, 6\}$

$A \cap B = \emptyset$



Set Cardinality

- For finite sets

  $A = \{ 2, 5, 7 \}$

  $|A| = 3$

Powersets

A powerset is a set of sets

$S = \{ a, b, c \}$

Powerset of S = the set of all the subsets of S

$2^S = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$

Observation: $| 2^S | = 2^{|S|}$ $\quad (8 = 2^3)$

Cartesian Product

$A = \{ 2, 4 \}$ $B = \{ 2, 3, 5 \}$

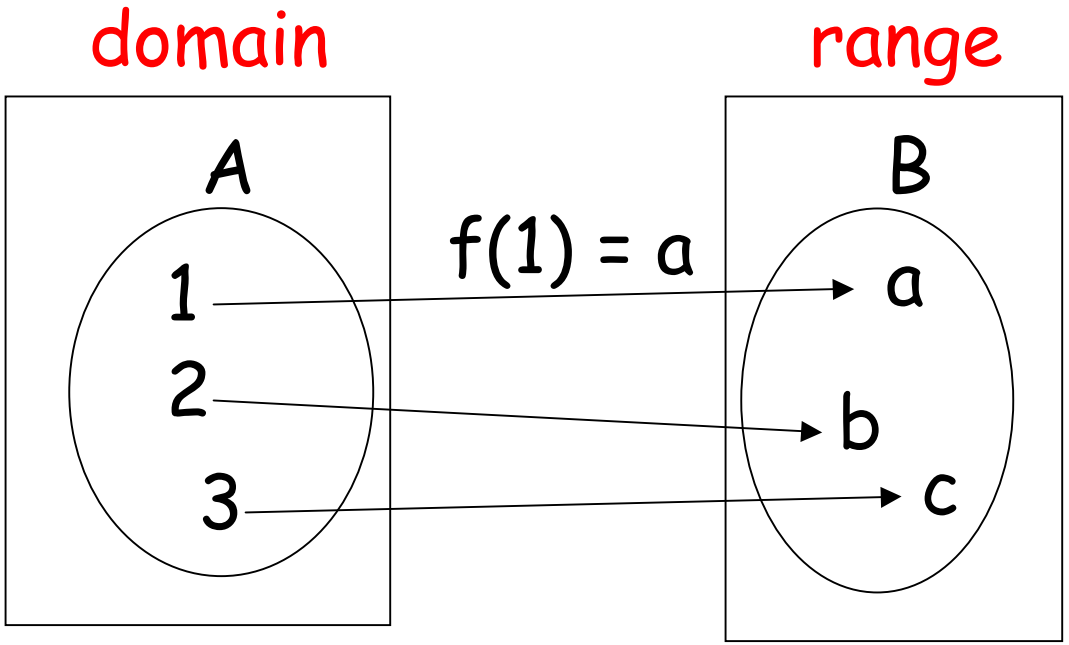
$A \times B = \{ (2, 2), (2, 3), (2, 5), (4, 2), (4, 3), (4, 4) \}$

$|A \times B| = |A| \, |B|$

Generalizes to more than two sets

$A \times B \times \ldots \times Z$

FUNCTIONS



f : A -> B

If A = domain

then f is a total function

otherwise f is a partial function

RELATIONS

$R = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$

$x_i \; R \; y_i$

e. g. if R = '>': $2 > 1$, $3 > 2$, $3 > 1$

In relations x_i can be repeated

Equivalence Relations

- Reflexive: $x \; R \; x$
- Symmetric: $x \; R \; y \Rightarrow y \; R \; x$
- Transitive: $x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$

Example: R = '='

- $x = x$
- $x = y \Rightarrow y = x$
- $x = y$ and $y = z \Rightarrow x = z$

Equivalence Classes

For equivalence relation R

equivalence class of $x = \{y : x \, R \, y\}$

Example:

$R = \{ (1, 1), (2, 2), (1, 2), (2, 1),$

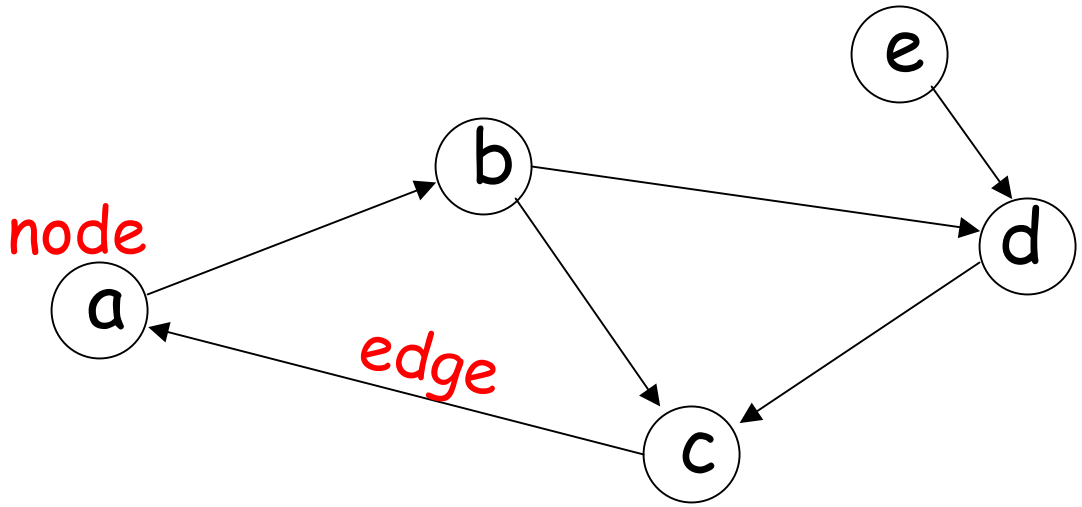
$(3, 3), (4, 4), (3, 4), (4, 3) \}$

Equivalence class of $1 = \{1, 2\}$

Equivalence class of $3 = \{3, 4\}$

GRAPHS

A directed graph



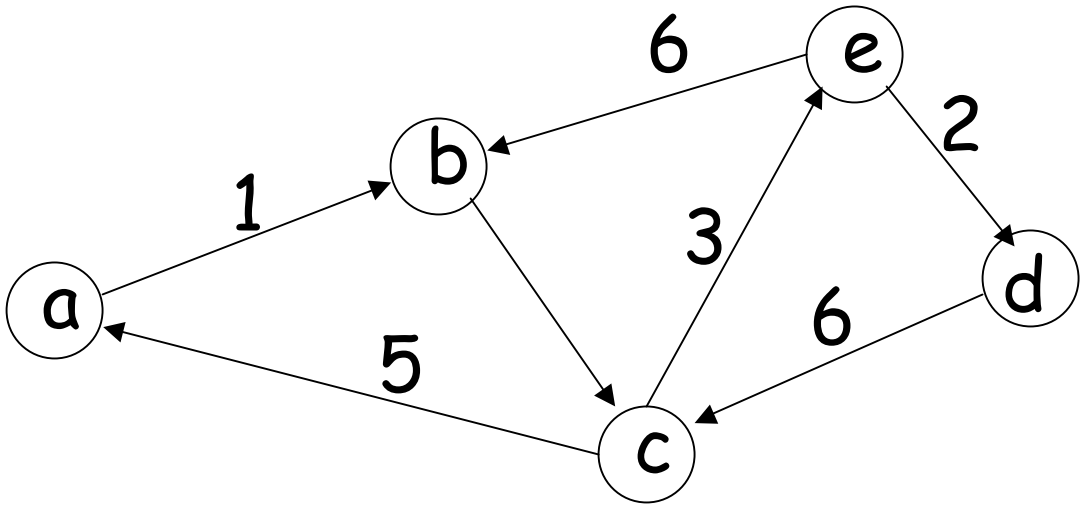
- Nodes (Vertices)

$$V = \{ a, b, c, d, e \}$$

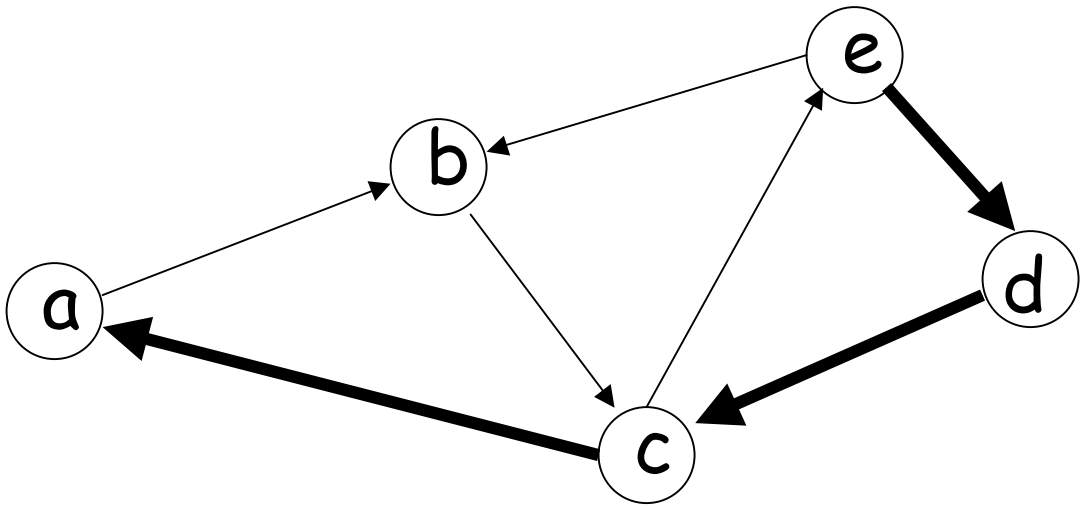
- Edges

$$E = \{ (a, b), (b, c), (c, a), (b, d), (d, c), (e, d) \}$$

Labeled Graph



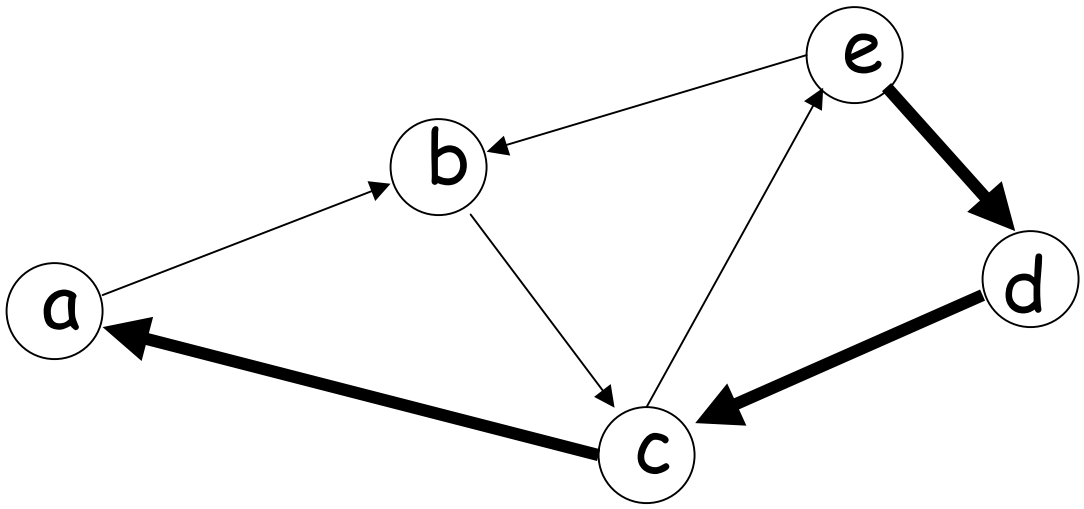
Walk



Walk is a sequence of adjacent edges

(e, d), (d, c), (c, a)

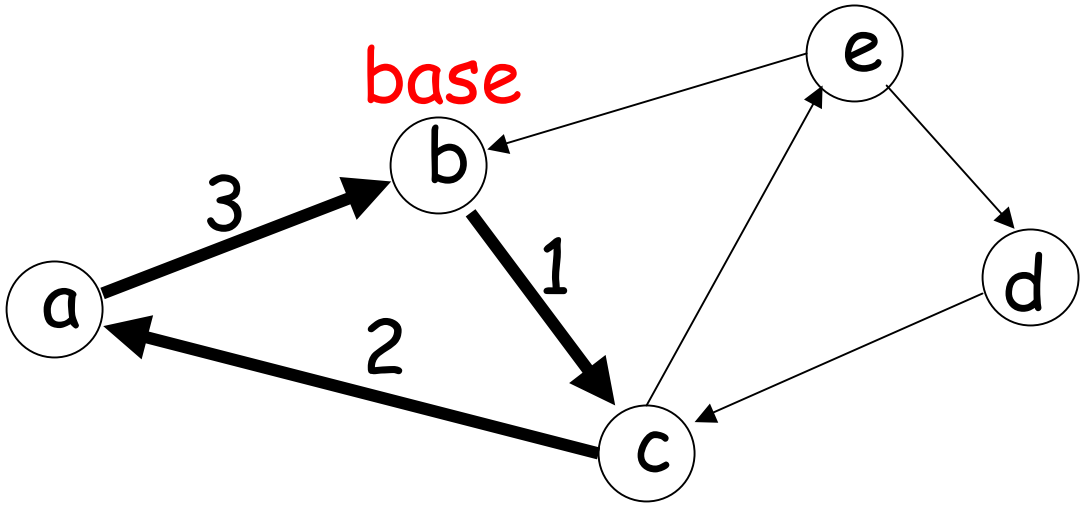
Path



Path is a walk where no edge is repeated

Simple path: no node is repeated

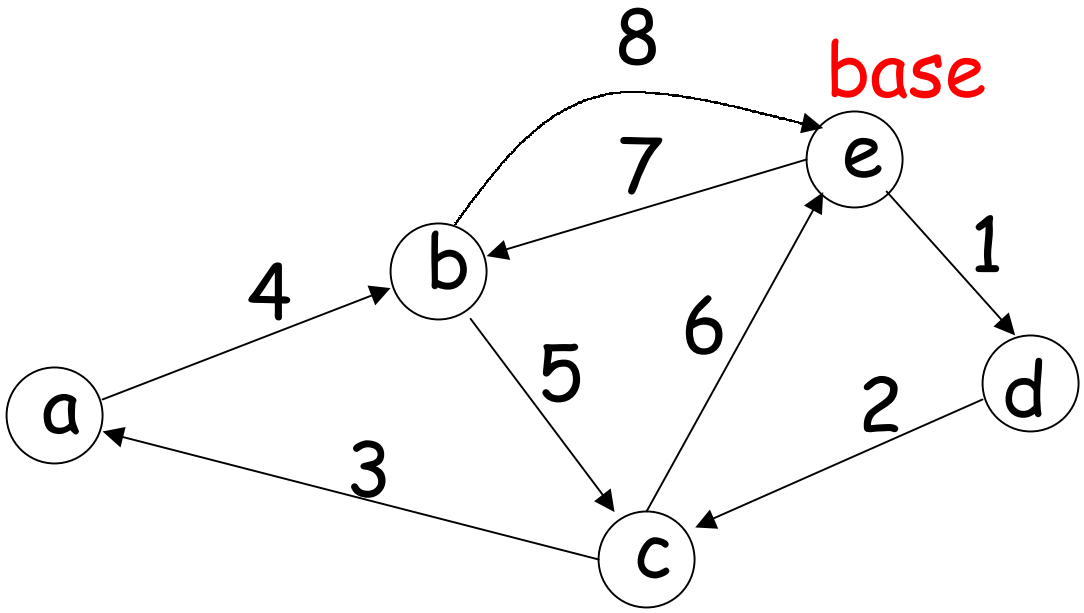
Cycle



Cycle: a walk from a node (base) to itself

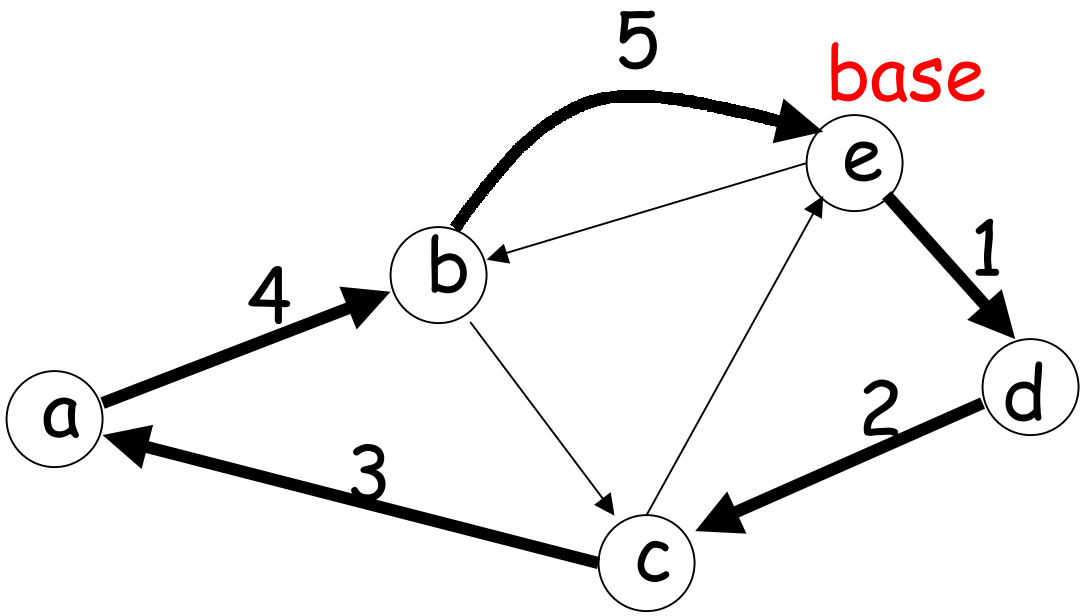
Simple cycle: only the base node is repeated

Euler Tour



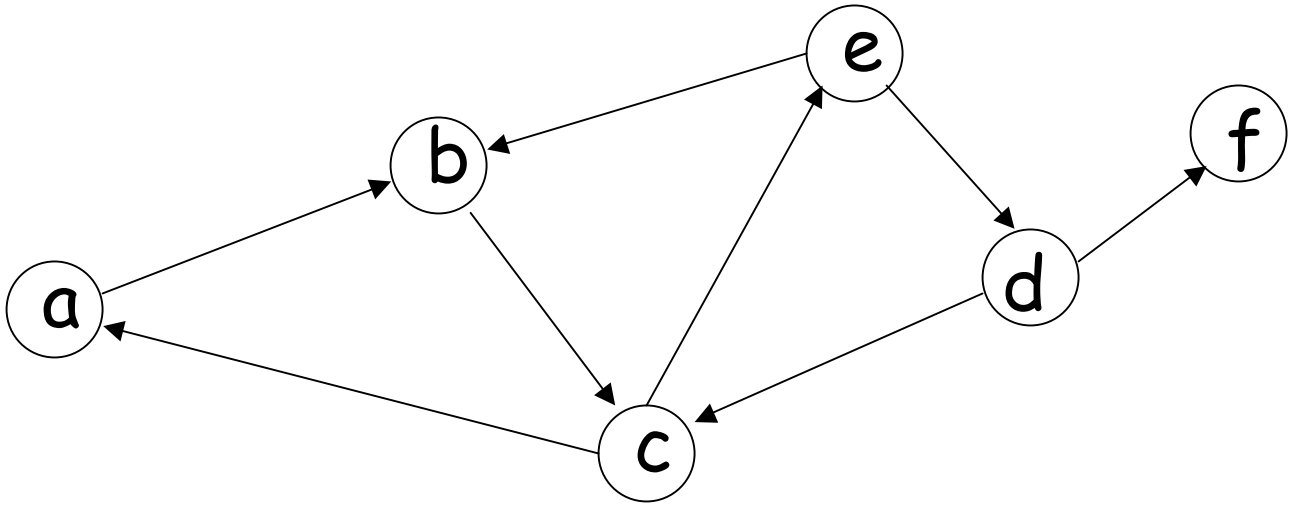
A cycle that contains each edge once

Hamiltonian Cycle

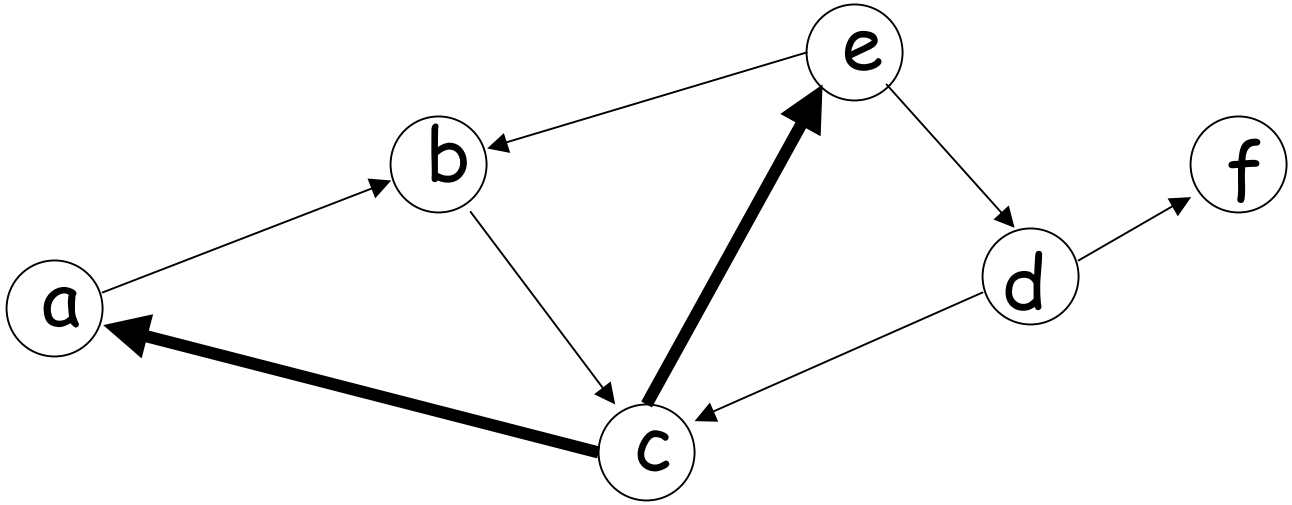


A simple cycle that contains all nodes

Finding All Simple Paths



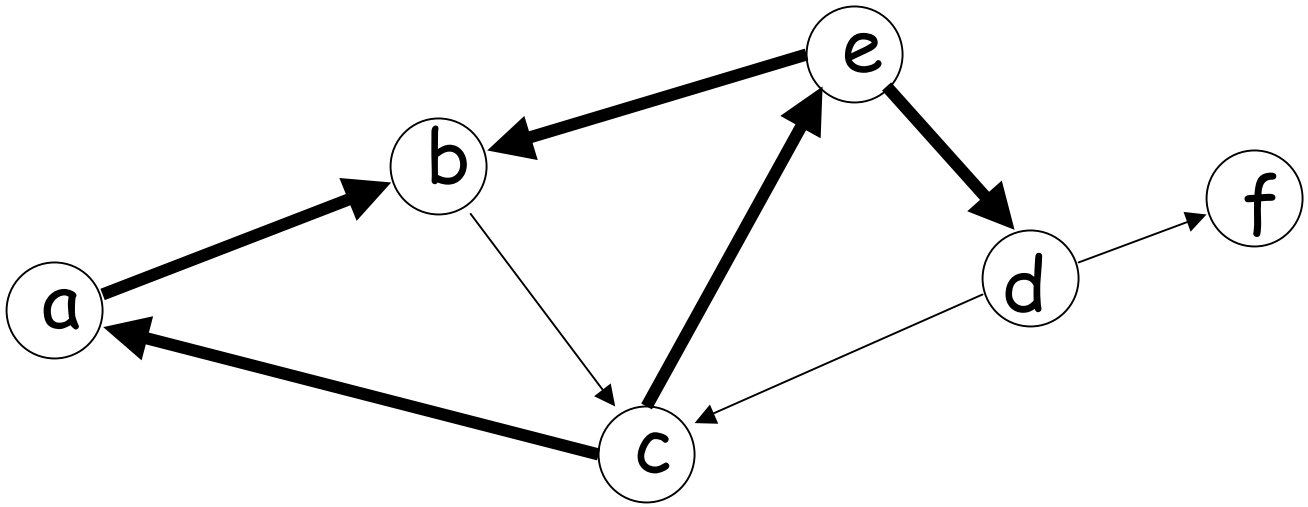
Step 1



(c, a)

(c, e)

Step 2



(c, a)

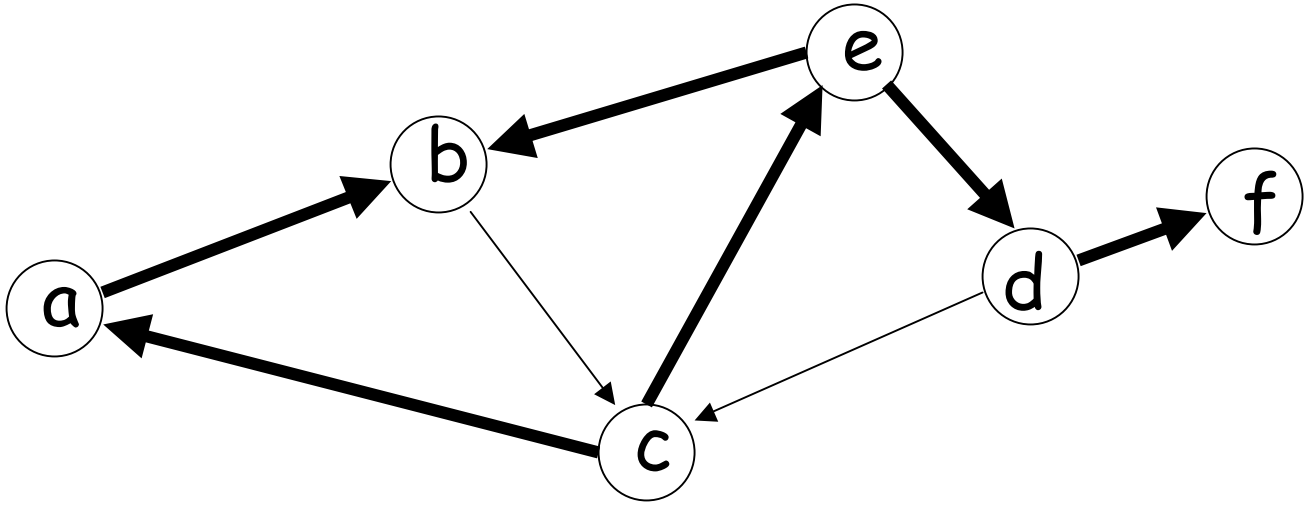
(c, a), (a, b)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 3



(c, a)

(c, a), (a, b)

(c, e)

(c, e), (e, b)

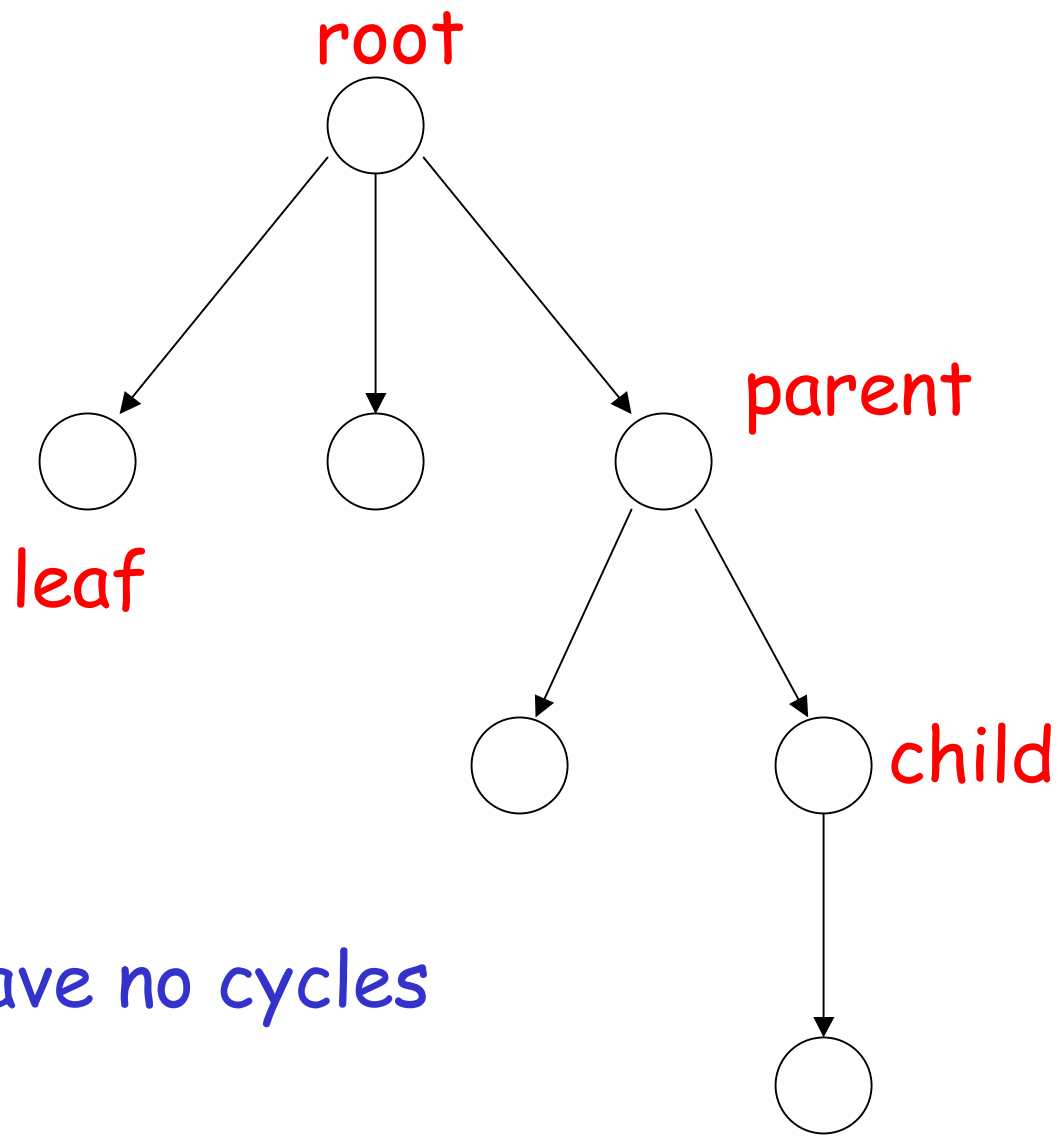
(c, e), (e, d)

(c, e), (e, d), (d, f)

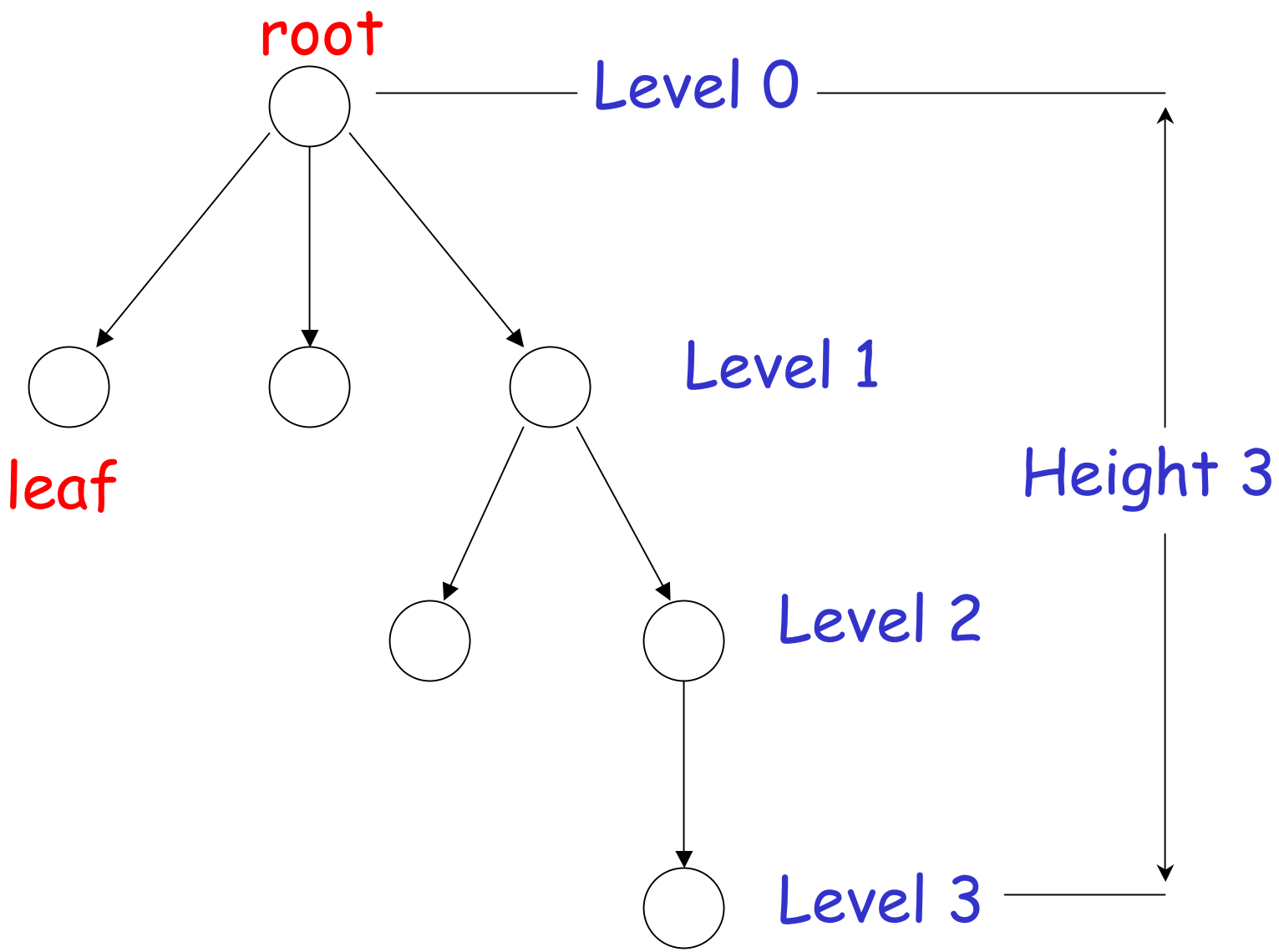
Repeat the same

for each starting node

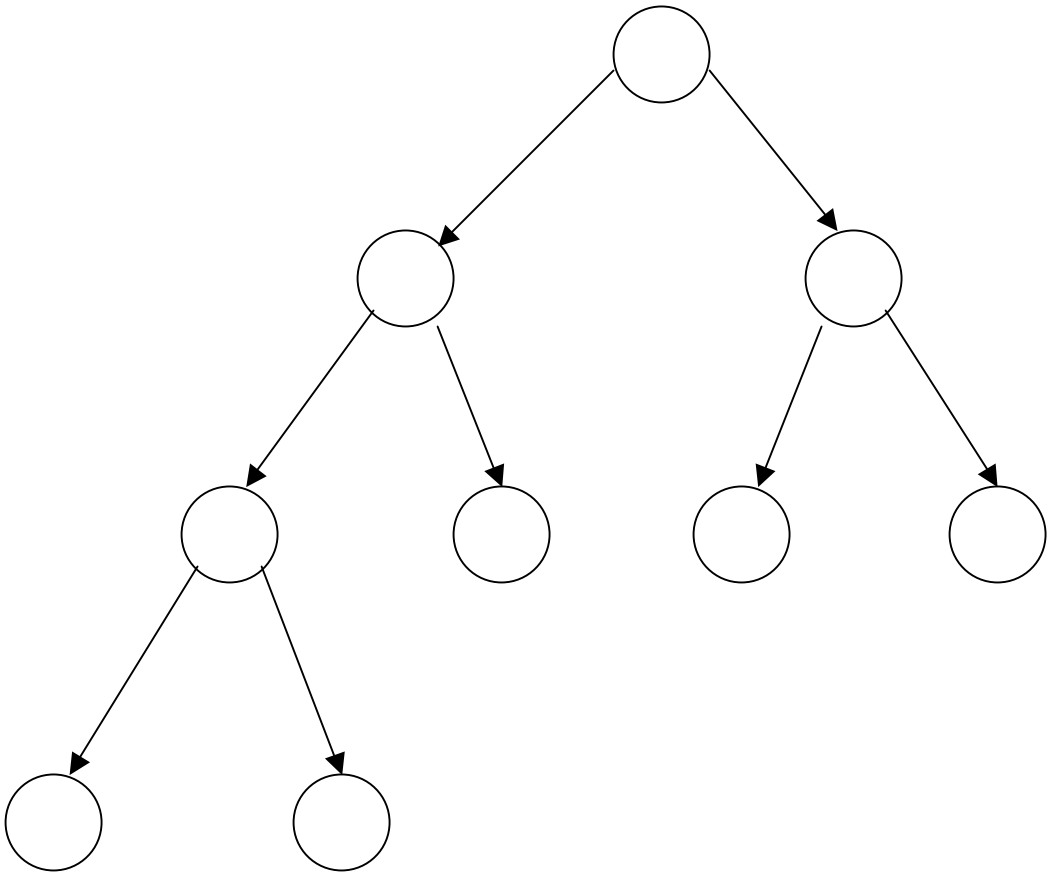
Trees



Trees have no cycles


root
Level 0
Level 1
leaf
Height 3
Level 2
Level 3

Binary Trees



PROOF TECHNIQUES

- Proof by induction

- Proof by contradiction

Induction

We have statements $P_1, P_2, P_3, \ldots$

If we know

- for some k that $P_1, P_2, \ldots, P_k$ are true
- for any n >= k that

$P_1, P_2, \ldots, P_n$ imply P_{n+1}

Then

Every P_i is true

Proof by Induction

- Inductive basis

  Find $P_1, P_2, \ldots, P_k$ which are true

- Inductive hypothesis

  Let's assume $P_1, P_2, \ldots, P_n$ are true,

  for any n >= k

- Inductive step

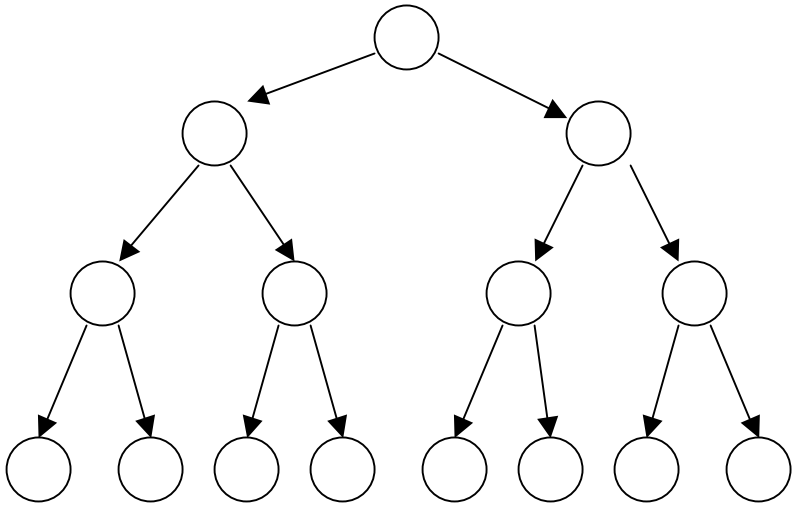
  Show that P_{n+1} is true

Example

Theorem: A binary tree of height n has at most 2^n leaves.

Proof:

let l(i) be the number of leaves at level i



l(0) = 1

l(3) = 8

We want to show: $l(i) <= 2^i$

- Inductive basis

  $l(0) = 1$ (the root node)

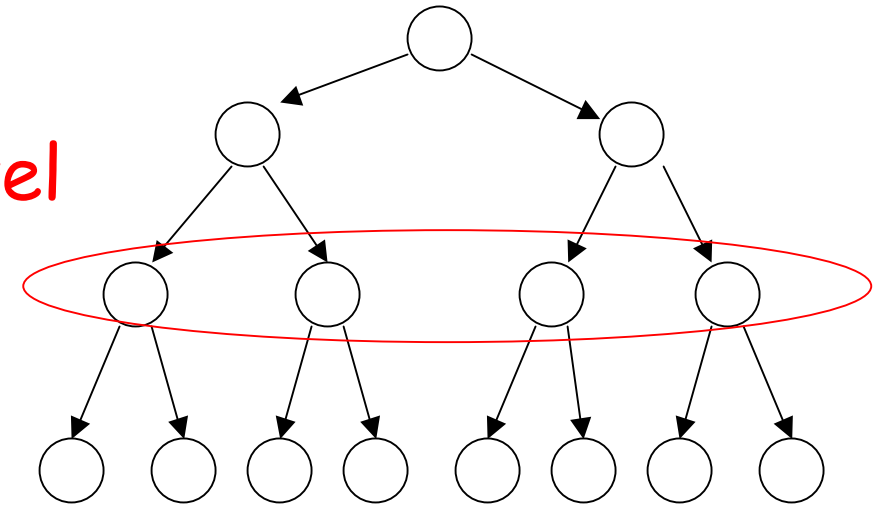
- Inductive hypothesis

  Let's assume $l(i) <= 2^i$ for all $i = 0, 1, \ldots, n$

- Induction step

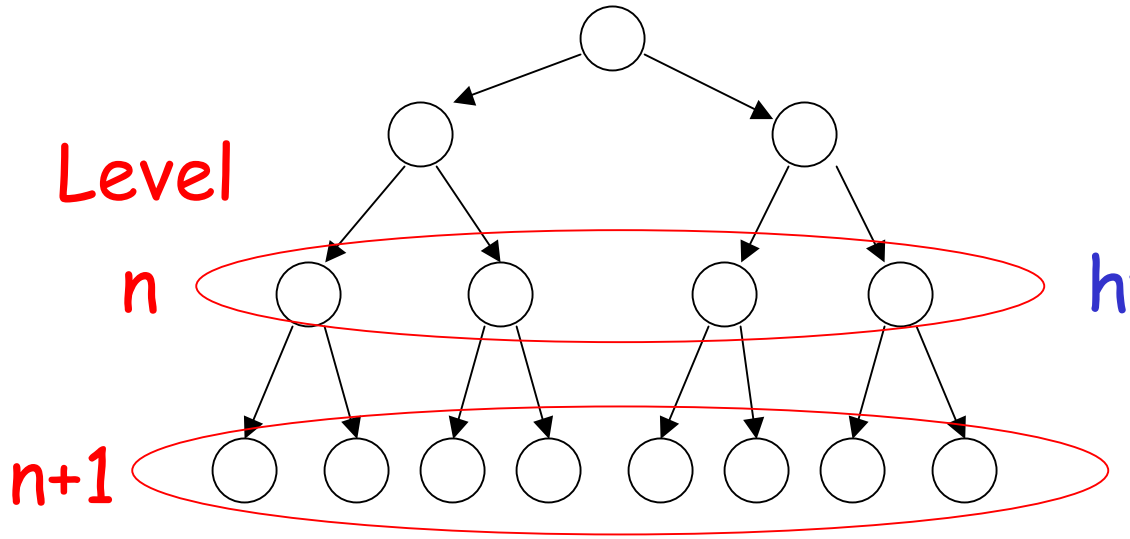
  we need to show that $l(n + 1) <= 2^{n+1}$

Induction Step



hypothesis: $l(n) <= 2^n$

Induction Step



hypothesis: $l(n) <= 2^n$

$$l(n+1) <= 2 * l(n) <= 2 * 2^n = 2^{n+1}$$

Remark

Recursion is another thing

Example of recursive function:

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 1, \quad f(1) = 1$$

Proof by Contradiction

We want to prove that a statement P is true

- we assume that P is false
- then we arrive at an incorrect conclusion
- therefore, statement P must be true

Example

Theorem: $\sqrt{2}$ is not rational

Proof:

Assume by contradiction that it is rational

$\sqrt{2}$ = n/m

n and m have no common factors

We will show that this is impossible

$\sqrt{2} = n/m \Rightarrow 2m^2 = n^2$

Therefore, n^2 is even $\Rightarrow$ n is even

$n = 2k$

$2m^2 = 4k^2 \Rightarrow m^2 = 2k^2 \Rightarrow$ m is even

$m = 2p$

Thus, m and n have common factor 2

Contradiction!