Solutions to Homework 1

Problem 1.

We prove the claim by induction on the height n.

- **Basis Case.** For $n = 0$ the tree consists only from one node. Therefore, the number of nodes in the tree is

$$2^{n+1} - 1 = 2^{0+1} - 1 = 2 - 1 = 1,$$

  as needed.

- **Inductive Hypothesis.** Let's assume that for any binary tree with height up to n, where $n \geq 0$, the number of nodes in the tree is at most $2^{n+1} - 1$.

- **Inductive Step.** We will prove that the claim holds for any binary tree of height $n + 1$. Namely, we will prove that the number of nodes in the tree is at most $2^{(n+1)+1} - 1$.

  Any binary tree of height $n + 1$ can be decomposed into two subtrees each of height n and a root node as shown in Figure 1. Since each subtree has height n, we can apply the inductive hypothesis to each subtree and the number of nodes in a subtree is at most $2^{n+1} - 1$. Summing the nodes from the two subtrees and the root node we have that the total number of nodes is at most

$$2 \cdot (2^{n+1} - 1) + 1 = 2^{n+2} - 2 + 1 = 2^{(n+1)+1} - 1,$$

  as needed.

Problem 2.

We prove the claim by induction on n.

- **Basis Case.** For $n = 4$ we have $n! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, and $2^4 = 16$, and thus $2^4 < n!$, as needed.

- **Inductive Hypothesis.** Let's assume that the claim holds for any integer up to n, where $n \geq 4$. Namely it holds that $2^n < n!$.

- **Inductive Step.** We will prove that the claim holds for integer $n + 1$ (where $n \geq 4$). Namely, we will prove that $2^{n+1} < (n + 1)!$.

  We can write $2^{n+1} = 2 \cdot 2^n$. By the inductive hypothesis we have that $2^n < n!$. Subsequently, $2 \cdot 2^n < 2 \cdot n!$. Since $2 < n$ we obtain $2 \cdot 2^n < n \cdot n!$. By the definition of the factorial operation we have that $(n+1)! = (n+1) \cdot n! > n \cdot n!$. Therefore, $2 \cdot 2^n < (n + 1)!$, which implies that $2^{n+1} < (n + 1)!$, as needed.

Problem 3.

We are given that for any symbol a, it holds

$$a^R = a, \tag{1}$$

and for any string u and symbol a, it holds

$$(ua)^R = au^R. \tag{2}$$

We want to prove that for any strings u and v it holds that

$$(uv)^R = v^R u^R.$$

We will prove the claim by induction on $|v|$, the length of the string v.

- **Basis Case.** We have $|v| = 1$, and thus v is only one symbol e.g. $v = a$. Threfore, $(uv)^R = (ua)^R$. By Equation 2, we have $(ua)^R = au^R$, and by Equation 1, we have $au^R = a^R u^R$. Since $a^R = v^R$, we obtain

  $$(uv)^R = (ua)^R = a^R u^R = v^R u^R,$$

  as needed.

- **Inductive Hypothesis.** Let's assume that the claim holds for any string v of length at most n, (in other words, $|v| \leq n$). Namely, for any such string it holds

  $$(uv)^R = v^R u^R.$$

- **Inductive Step.** We will prove the claim for any string v of length equal to $n + 1$ (in other words, $|v| = n + 1$). Namely, we will prove that

  $$(uv)^R = v^R u^R.$$

  Since $|v| = n + 1$, we can write v as the concatenation of one string, e.g. w, and a symbol, e.g. a, so that $v = wa$, where the string w has length $|w| = n$. We have now, that

  $$\begin{aligned}
  (uv)^R &= (uwa)^R \\
  &= a^R(uw^R) \\
  &\quad \text{(By Equation 2 applied on } uw \text{ and } a\text{)} \\
  &= a(uw)^R. \\
  &\quad \text{(By Equation 1)}
  \end{aligned}$$

  Since the length of w is n, we can apply the inductive hypothesis for the string w and we obtain

  $$(uw)^R = w^R u^R.$$

Subsequently,

$$
\begin{aligned}
(uv)^R &= a(uw)^R \\
&= aw^R u^R \\
&\quad \text{(By the inductive hypothesis on } w\text{)} \\
&= (wa)^R u^R \\
&\quad \text{(By Equation 2 applied on } w \text{ and } a\text{)} \\
&= (v)^R u^R \\
&= v^R u^R,
\end{aligned}
$$

as needed.

Problem 4.

We prove the two parts of the problem.

- First we show that if $w \in L_1(L_2 \cap L_3)$ then $w \in L_1L_2 \cap L_1L_3$.

  Since $w \in L_1(L_2 \cap L_3)$, there must be two strings u and v, such that $u \in L_1$ and $v \in L_2 \cap L_3$, so that w can be written as the concatentation of u and v, namely $w = uv$.

  Since $v \in L_2 \cap L_3$, it must be that $v \in L_2$ and $v \in L_3$. Furthermore, since $u \in L_1$, we get that $uv \in L_1L_2$ and $uv \in L_1L_3$. Subsequently, $uv \in L_1L_2 \cap L_1L_3$, and thus, $w \in L_1L_2 \cap L_1L_3$, as needed.

- We want to find languages L_1, L_2, L_3, for which there is a w such that if $w \in L_1L_2 \cap L_1L_3$ then $w \notin L_1(L_2 \cap L_3)$.

  Take

$$
\begin{aligned}
L_1 &= \{a, ab\} \\
L_2 &= \{\lambda\} \\
L_3 &= \{b\}.
\end{aligned}
$$

  We get,

$$
\begin{aligned}
L_1L_2 &= \{a, ab\}\{\lambda\} = \{a, ab\} \\
L_1L_3 &= \{a, ab\}\{b\} = \{ab, abb\},
\end{aligned}
$$

  Subsequenlty,

$$L_1L_2 \cap L_1L_3 = \{a, ab\} \cap \{ab, abb\} = \{ab\}.$$

  Moreover,

$$L_2 \cap L_3 = \{\lambda\} \cap \{b\} = \emptyset,$$

and thus

$$L_1(L_2 \cap L_3) = \{a, ab\}\emptyset = \emptyset.$$

Take now

$$w = ab.$$

We have

$$ab \in L_1L_2 \cap L_1L_3 = \{ab\}$$

and

$$ab \notin L_1(L_2 \cap L_3) = \emptyset,$$

as needed.

Problem 5.

See Figures 2, 3, 4, and 5.

Figure 1: Decomposition of a binary tree to two subtrees.

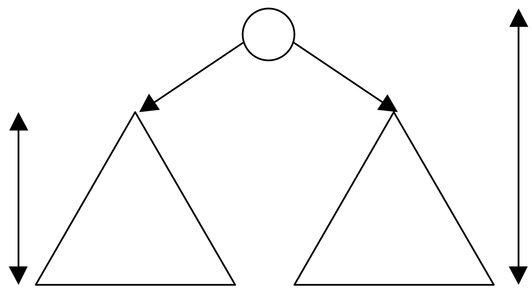

Figure 2: Part (a)

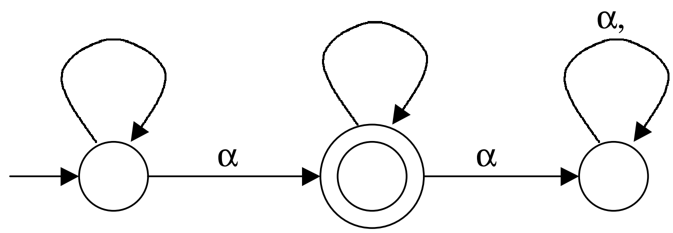

Figure 3: Part (b)

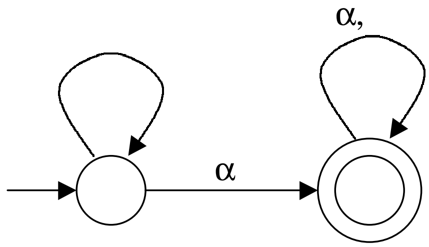

Figure 4: Part (c)

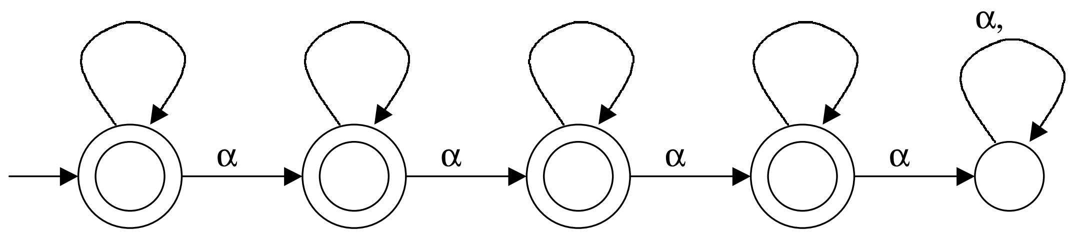

Figure 5: Part (d)